

Mr's Henry Jones
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C. A.

LOVELL'S SERIES OF SCHOOL BOOKS.

NATIONAL
ARITHMETIC,

IN

THEORY AND PRACTICE;

DESIGNED FOR THE USE OF

CANADIAN SCHOOLS.

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PHILOSOPHY IN THE NORMAL SCHOOL FOR UPPER CANADA.

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PREFACE.

In preparing the following work (undertaken at the suggestion of the Chief Superintendent of Education for Upper Canada), it has been the constant aim of the author to present it to Canadian teachers and students as a thoroughly reliable Treatise on the Theory and Practice of Numbers, and as an Arithmetic, in some degree, commensurate with the higher qualifications of teachers and the improved methods of instruction now generally found in our schools.

The Arithmetic now offered to the public is based upon the Irish National Treatise;—in fact, it was at first intended merely to adapt that work to the decimal currency, and to abbreviate the somewhat tedious reasons there given for the various rules. So many alterations and improvements suggested themselves, however, that the original design was speedily abandoned, and, with the exception of the first ten or fifteen pages, which are taken entire from the work in question, the Treatise, as at present issued, is, in all essential respects, an entirely new book. Nevertheless, as it was the sole object of the author to prepare a *complete* text-book on the subject of Arithmetic, he has not hesitated to adopt whatever he considered good, either in the Irish National or in the numerous other excellent works on the subject.

By far the greater number of the problems are original; and it is hoped that the practical manner in which many of them are put, will tend to render the study of Arithmetic more interesting and useful than it has hitherto been. It will be observed, that a thorough series of review examples has been given at the close of each of the sections up to the seventh, and a very extensive set at the end of the book. This is deemed an important feature in the present work, as in some degree insisting upon that careful revision of what has been learned from time to time, without which, the pupil arrives at the end of the book with all the rules and principles so confounded with one another, as to render his knowledge in a great measure worthless.

Since the only difference between simple and denominate numbers is that the one increase and decrease according to the scale of tens and the other according to different scales, there is no reason why the rules relating to them should be separated; and therefore in the following pages no distinction is made between simple and compound rules. A somewhat extended

experience has convinced the author that, except to the merest beginners, the science of Arithmetic is more successfully presented by this than by the ordinary method of making the pupil learn one set of rules for simple numbers and a completely different set for compound numbers.

It will be observed that towards the end of the Treatise the rules are mainly deduced algebraically. Some teachers may not, at first, be disposed to regard this as an improvement, but it was not adopted until after careful deliberation and consultation with many of the most successful teachers of Arithmetic in the Province. It is generally conceded that a pupil should commence, in some sort, the study of Algebra as soon as he has progressed through Proportion in Arithmetic. In schools in which this view is adopted by the teacher, no difficulty can be experienced, as, even in the deduction of the rules, the algebraic principles used are of the simplest possible character.

As some teachers, however, prefer always giving the rule in a purely arithmetical form, this has invariably been appended in all the cases usually treated of in Common Arithmetic.

With regard generally to algebraic formulas, it may be further remarked, that an algebraic formula is simply the most abbreviated form in which it is possible to express a rule or principle. Once the pupil is properly taught their use, he is in a manner independent of mere memory, since from a very few general principles he is able, without any reference to a text-book, to deduce for himself the whole series of rules for Simple and Compound Interest, Discount, Annuities, Progression, and Position. Even when the pupil is merely required to commit the rules to memory, it is obvious that he can do so much more readily when they are given to him in the shape of algebraic formulas than in long worded paragraphs. Let any one, for instance, compare the work necessary for committing the eleven rules for simple interest with that required to commit the corresponding formulas, and the result will be a thorough conviction of the superiority of the latter mode of giving the rules. In short, every experienced teacher will admit, that even while the pupil remains at school it is next to impossible to make him remember all the different rules for Interest, Progression, and Annuities; and that directly he leaves the school to enter upon the business of life, these rules are either altogether forgotten or are so confounded with one another as to become mere useless mental lumber. After many years' trial, the author is persuaded that the only successful mode of treating the rules in question, is to enable the pupil to deduce them algebraically and then to interpret and apply the resulting formulas.

The attention of the teacher is respectfully directed to the Recapitulation at the end of the first section, where, it is thought,

the definition and essential principles of Notation and Numeration are so concisely worded that they may be advantageously committed to memory by the pupil.

The examination questions throughout the work have been carefully prepared, and are designed both to enable the self-taught student to test, at each section, the extent and thoroughness of his knowledge of the principles therein contained, and also to guide the pupil as to what principles and definitions are of such importance that they require to be committed to memory. This latter object is further secured by the arrangement of type, —all the definitions and leading principles being printed in large type, the explanations, reasons, and remarks in small type, and the problems in a size intermediate to the two.

Great pains have been taken to render the wording of the rules as perfect as possible; and it will be observed that, in order to catch the eye when glancing over the page, they are invariably printed in Italics.

It is believed that the sections on Proportion, Fractions, Interest, &c., contain a larger amount of information and a better selection of examples than are commonly given; and that the section on the Properties of Numbers and the different scales of notation will tend very materially to enlarge the pupil's acquaintance with the general principles of the science of Arithmetic.

Although the Preface is not the proper place for discussing methods of teaching Arithmetic, the author cannot refrain from urging upon his fellow-teachers the following points:

1st. The pupil should be thoroughly drilled upon the use of the signs and symbols of Arithmetic, because these constitute the language proper to the subject.

2nd. He should be required to commit to memory all the essential definitions, and also the tables of money, weights, and measures. The teacher would do well to examine his pupils on these tables once a month or oftener, since if the pupil has to turn back to his book for each table as it is required, it is not to be expected that his progress will be very rapid or thorough. It may be fairly questioned, whether more than half the difficulty and obscurity that cling to the subject of Arithmetic does not arise from the fact that the pupil is not familiar with the signs, the tables, and the principles of notation.

3rd. The teacher should give his class, from time to time, questions of his own construction, either to solve at home or as ordinary school-room work, and the pupils should be encouraged and required to write questions themselves under each rule. This is an important exercise, and no teacher who once adopts it will ever throw it aside.

4th. In all operations in which there are both multiplication and division, the pupil should be taught to first indicate the processes by their appropriate signs and then cancel as far as possible.

5th. The teacher is respectfully reminded, that without frequent and thorough reviews there can be no real progress. Experience has shown that from one-third to one-half of the time devoted to Arithmetic can be profitably devoted to revision and recapitulation.

6th. The teacher should require from his pupil the absolutely correct answer to each question. '*Near enough*' is productive of great mischief to the pupil, as it encourages a habit of such carelessness in his operations, that no confidence can be placed on his results. It is not enough that the pupil understands the principles,—although this of course is important. It is possible so to train the pupil that his operations in Arithmetic shall be at once rapid and accurate, and this should be the aim of the teacher.

Toronto, December, 1859.

PREFACE TO THE SECOND EDITION.

The Author embraces the opportunity afforded by the issue of a Second Edition, both to thank his fellow-teachers in Canada for the kind and flattering reception they have given his work, and to offer a few words of explanation on what, as far as he can learn, is the only feature that does not meet with very general approval. He refers to the union of the Compound with the Simple Rules. It has been objected to the arrangement adopted in the National Arithmetic, that a pupil must know the Simple Rules before he can work problems in Reduction or in the Compound Rules. Now this is undoubtedly true, and would be a fatal objection to any such arrangement in an Elementary or Primary Arithmetic. The National is, however, an *advanced* or *second* book on arithmetic, and the pupil is assumed to have progressed through an elementary text-book before he enters it. If the National Arithmetic were designed for beginners, where would be the necessity for a First or Elementary book on Arithmetic? The objections have arisen altogether from a misconception of the design of the book. The pupil is supposed to have worked through some elementary text-book on arithmetic, and to have acquired a certain amount of practical skill in arithmetical operations. He then enters the National, and, in progressing through it, not only meets with additional and more advanced practical exercises, but also learns the reasons and the mutual relations of the several rules. In the Elementary he is taught how to multiply an abstract by an abstract number, or an applicate by an abstract number. In the National he is shown that these operations, though differing in detail, are essentially the same in principle; and he is thus enabled to generalize and classify.

Another objection urged is, that if the National Arithmetic be designed for a second book on the science, the simple problems given at the commencement of each rule, and indeed the earlier rules themselves, should not be inserted. This is also a mistake. The object has been to exhibit a gradual progression from the simple to the more difficult,—to show that the most simple and the most complicated problems depend essentially upon the same principles. Indeed, were the National Arithmetic intended merely as a second *practical* work on arithmetic, three-fourths of it might have been omitted, and nothing given but the two or three rules omitted in the Elementary.

PUBLISHER'S ADVERTISEMENT.

In the first edition of this work some few typographical errors escaped notice.

The rapid sale of the book having necessitated another edition, these errors have been corrected; and the Publisher has the satisfaction, in issuing this edition, of stating, that it has undergone the most rigid and careful revision at the hands of the Author.

The next edition will be printed from Stereotype Plates.

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SIGNS USED IN THIS TREATISE.

+ the sign of addition ; as $5+7$, or 5 to be added to 7.
 — the sign of subtraction ; as $4-3$, or 3 to be subtracted from 4.

× the sign of multiplication ; as 8×9 , or 8 to be multiplied by 9.

÷ the sign of division ; as $18\div 6$, or 18 to be divided by 6.

() which is used to show that all the quantities united by it are to be considered as but one. Thus $(4+3-7\times 6)$ means 4 to be added to 3, 7 to be taken from the sum, and 6 to be multiplied into the remainder. The latter is equivalent to the *whole* quantity within the brackets.

= the sign of equality ; as $5+6=11$, or 5 added to 6, is equal to 11.

$\frac{2}{3} > \frac{1}{2}$, and $\frac{2}{3} < \frac{5}{6}$, mean that $\frac{2}{3}$ is greater than $\frac{1}{2}$, and that $\frac{2}{3}$ is less than $\frac{5}{6}$.

: is the sign of ratio or relation ; thus $5 : 6$, means the ratio of 5 to 6, and is read 5 is to 6.

:: indicates the equality of ratios ; thus $5:10::7:14$, means that there is the same relation between 5 and 10 as between 7 and 14 ; and is read 5 is to 10 as 7 is to 14.

√ the radical sign. By itself, it is the sign of the square root ; as $\sqrt{5}$, which is the same as $5^{\frac{1}{2}}$, the square root of 5. $\sqrt[3]{3}$, is the cube root of 3, or $3^{\frac{1}{3}}$. $\sqrt[7]{4}$ is the 7th root of 4, or $4^{\frac{1}{7}}$, &c.

EXAMPLE. [$\sqrt{\{(8-3+7)\times 4\div 6\}+31} \times \sqrt[3]{9\div 10^{\frac{1}{2}}} \times 5^2=556\cdot 3$, &c., may be read thus : take 3 from 8, add 7 to the difference, multiply the sum by 4, divide the product by 6, take the square root of the quotient and to it add 31, then multiply the sum by the cube root of 9, divide the product by the square root of 10, multiply the quotient by the square of 5, and the product will be equal to $556\cdot 3$, &c.

These signs are *fully* explained in their proper places.

ARITHMETIC.

SECTION I.

DEFINITIONS.

1. Science is a collection of the general *principles* or leading *truths* relating to any branch of knowledge, arranged in systematic order so as to be readily remembered, referred to, and applied.

2. Art is a collection of *rules* serving to facilitate the performance of certain operations. The *rules* of Art are based upon the *principles* of Science.

3. Arithmetic is both a *Science* and an *Art*.

4. As a Science, Arithmetic treats of the nature and properties of numbers; as an Art, it teaches the mode of applying this knowledge to practical purposes. The former may be called Theoretical, and the latter Practical Arithmetic. To Practical Arithmetic belong all the *operations* we perform upon numbers, as addition, subtraction, multiplication, division, the extraction of roots, &c. The discussion of the *principles* upon which these operations are founded, constitutes the theory of Arithmetic.

5. Any single thing, as a horse, an apple, a day, an inch, is called a *unit* or *one*.

6. Numbers are expressions for one or more units. Thus, the *words one, two, three, four, five, &c.*, or the *characters 1, 2, 3, 4, 5, &c.*, are expressions by which we indicate how many single things or units are to be taken.

7. Numbers are divided into two classes.

1. Abstract numbers.

2. Applicate, Concrete, or Denominate numbers.

8. If the units referred to by a number have reference to *particular* objects, as *seven days, nine inches, &c.*, it is called an *applied, applicate, concrete, or denominate number*. If the units represented by a number have *no* reference to any particular object, as when we say *twice eight are sixteen, or seven and two are nine*, it is called an *Abstract number*.

NOTATION AND NUMERATION.

9. To avail ourselves of the properties of numbers, we must be able both to form an idea of them ourselves; and to convey this idea to others by spoken and by written language—that is, by the voice, and by characters.

The expression of number by characters, is called *notation*; the reading of these, *numeration*. Notation, therefore, and numeration, bear the same relation to each other as *writing* and *reading*, and, though often confounded, they are in reality perfectly distinct.

10. It is obvious that, for the purposes of Arithmetic, we require the power of designating all possible numbers; it is equally obvious that we cannot give a different name or character to each, as their variety is boundless. We must, therefore, by some means or another, make a limited system of words and signs suffice to express an unlimited amount of numerical quantities. With what beautiful simplicity and clearness this is effected, we shall better understand presently.

11. Two modes of attaining such an object present themselves: the one, that of *combining* words or characters already in use, to indicate new quantities; the other, that of representing a variety of different quantities by a *single* word or character, the danger of mistake at the same time being prevented. The Romans simplified their system of notation by adopting the principle of *combination*; but the still greater perfection of ours is due also to the expression of many numbers by the *same* character.

12. It will be useful, and not at all difficult, to explain to the pupil the mode by which, as we may suppose, an idea of considerable numbers was originally acquired, and of which, indeed, although unconsciously, we still avail ourselves; we shall see, at the same time, how methods of simplifying both numeration and notation were naturally suggested.

Let us suppose no system of numbers to be as yet constructed, and that a heap, for example, of pebbles, is placed before us that we may discover their amount. If this is considerable, we cannot ascertain it by looking at them altogether, nor even by separately inspecting them; we must, therefore, have recourse to

that contrivance which the mind always uses when it desires to grasp what, taken as a whole, is too great for its powers. If we examine an extensive landscape, as the eye cannot take it all in at one view, we look successively at its different portions, and form our judgment on them in detail. We must act similarly with reference to large numbers: since we cannot comprehend them at a single glance, we must divide them into a sufficient number of parts, and, examining these in succession, acquire an indirect, but accurate idea of the whole. This process becomes by habit so rapid, that it seems, if carelessly observed, but one act, though it is made up of many: it is indispensable, whenever we desire to have a *clear idea* of numbers—which is not, however, every time they are mentioned.

13. Had we, then, to form for ourselves a numerical system, we should naturally divide the individuals to be reckoned into equal groups, each group consisting of some number quite within the limit of our comprehension; if the groups were few, our object would be attained without any further effort, since we should have acquired an accurate knowledge of the number of groups, and of the number of individuals in each group, and therefore a satisfactory, although indirect estimate of the whole.

We ought to remark, that different persons have very different limits to their perfect comprehension of number. The intelligent can conceive with ease a comparatively large one: there are savages so rude as to be incapable of forming an idea of one that is extremely small.

14. Let us call the *number* of individuals that we choose to constitute a group, the *ratio*; it is evident that the larger the ratio, the smaller the number of groups; and the smaller the ratio, the larger the number of groups.

15. If the groups into which we have divided the objects to be reckoned, exceed in amount that number of which we have a perfect idea, we must continue the process, and, considering the groups themselves as individuals, must form with them new groups of a higher order. We must thus proceed until the number of our highest group is sufficiently small.

16. The *ratio* used for groups of the second and higher orders, would naturally, but not necessarily, be the same as that adopted for the lowest; that is, if seven individuals constitute a group of the first order, we should probably make seven groups of the first order constitute a group of the second also; and so on.

17. It might, and very likely would happen, that we should not have so many objects as would *exactly* form a certain number of groups of the highest order—some of the next lower might be left. The same might occur in forming one or more of the other groups. We might, for example, in reckoning a heap of pebbles, have two groups of the fourth order, three of

the third, none of the second, five of the first, and seven individuals or simple units.

18. If we had made each of the first order of groups consist of ten pebbles, and each of the second order consist of ten of the first, each group of the third of ten of the second, and so on with the rest, we had selected the *decimal* system, or that which is not only used at present, but which was adopted by the Hebrews, Greeks, Romans, &c. It is remarkable that the language of every civilized nation gives names to the different groups of this, but not to those of any other numerical system. Its very general diffusion, even among rude and barbarous people, has most probably arisen from the habit of counting on the fingers, which is not altogether abandoned, even by us.

19. It was not indispensable that we should have used the same *ratio* for the groups of all the different orders. We might, for example, have made four pebbles form a group of the first order, twelve groups of the first order a group of the second, and twenty groups of the second a group of the third order.—In such a case we had adopted a system exactly like that to be found in the table of Sterling money, in which four farthings make a group of the order of *pence*, twelve pence a group of the order of *shillings*, twenty shillings a group of the order of *pounds*. While it must be admitted that the use of the same system for applicate, as for abstract numbers, would greatly simplify our arithmetical processes—as will be evident hereafter—a glance at the tables given further on, and those set down in treating of exchange, will show that a great variety of systems have actually been constructed.

20. When we use the same *ratio* for the groups of all the orders, we term it a *common ratio*. There appears to be no particular reason why *ten* should have been selected as a “common ratio” in the system of numbers ordinarily used, except that it was suggested, as already remarked, by the mode of counting on the fingers; and that it is neither so low as unnecessarily to increase the number of orders of groups, nor so high as to exceed the conception of any one for whom the system was intended. (See Section III.)

21. A system of numbers is called *binary*, *ternary*, *quaternary*, *quinary*, *senary*, *septenary*, *octenary*, *nonary*, *denary*, *undenary*, or *duodenary*, according as *two*, *three*, *four*, *five*, *six*, *seven*, *eight*, *nine*, *ten*, *eleven*, or *twelve*, is the *common ratio*. The *denary* and *duodenary* systems are more commonly known as the *decimal* and *duodecimal* systems. Ours is therefore a *decimal*, or *denary* system of numbers.

If the common ratio were sixty, it would be a *sexagesimal* system. Such a one was formerly used, and is still, to some extent, retained—as will be perceived by the tables hereafter given

for the measurement of arcs and angles, and of time. A duodecimal system would have twelve for its "common ratio"; a vigesimal, twenty, &c.

22. A little reflection will show that it was useless to give different names and characters to any numbers except to those which are less than that which constitutes the lowest group, and to the *different orders* of groups; because all possible numbers must consist of individuals, or of groups, or of both individuals and groups. In neither case would it be required to specify more than the number of individuals, and the number of each species of group, none of which numbers—as is evident—can be greater than the common ratio. This is precisely what we have done in our numerical system, except that we have formed the name of some of the groups by combining those already used. Thus, "tens of thousands," the group next higher than thousands, is designated by a combination of words already applied to express other groups—which tends still further to simplification.

23. *Arabic system of Notation* :—

| | <i>Names.</i> | <i>Characters.</i> |
|---|------------------------|--------------------|
| | One . . . | 1 |
| | Two . . . | 2 |
| | Three . . . | 3 |
| | Four . . . | 4 |
| Units of Comparison, or simple units, | Five . . . | 5 |
| | Six . . . | 6 |
| | Seven . . . | 7 |
| | Eight . . . | 8 |
| | Nine . . . | 9 |
| First group, or units of the second order, . | Ten . . . | 10 |
| Second group, or units of the third order, . | Hundred . . . | 100 |
| Third group, or units of the fourth order, . | Thousand . . . | 1,000 |
| Fourth group, or units of the fifth order, . | Ten Thousand . . . | 10,000 |
| Fifth group, or units of the sixth order, . | Hundred Thousand . . . | 100,000 |
| Sixth group, or units of the seventh order, . | Million . . . | 1,000,000 |

24. The characters which express the first nine numbers are the only ones used. They are called *digits*, from the custom of counting them on the fingers, already noticed,—“digitus” meaning in Latin a finger; and they have also been called *significant figures*, to distinguish them from the cipher, or 0, which has no value when standing alone, and which is used merely to give the digits their proper *position* with reference to the *decimal point*.

25. The *decimal point* is a point or dot used to indicate the position of the simple units.

The pupil will distinctly remember that the place where the “simple units” are to be found is that immediately to the left-hand of this point, which, if not expressed, is *supposed* to stand at the right-hand side of all the digits. Thus, in 468·76 the 8 expresses “simple units,” being to the left of the decimal point;

in 49 the 9 expresses "simple units," the decimal point being *understood* at the right of it.

26. We find by the table just given, that, after the first nine numbers, the same digits are constantly repeated, their positions with reference to the decimal point being, however, changed; that is, to indicate succeeding groups, the digit is moved, by means of a cipher, one place farther to the left. Any one of the digits may be used to express its respective number of any of the groups:—thus 8 would be eight "simple units"; 80, eight groups of the first order, or eight "tens" of simple units; 800, eight groups of the second, or units of the third order; and so on. We might use any of the digits with different groups; thus, for example, 5 for groups of the third order, 3 for those of the second, 7 for those of the first, and 8 for the "simple units," then the whole set down in full would be 5000, 300, 70, 8, or, for brevity's sake, 5378. For we *never* use a cipher, when the place it would occupy may be filled up by a digit; and it is evident that in 5378 the 378 keeps the 5 four places from the decimal point (*understood*), just as well as ciphers would have done; also the 78 keeps the 3 in the third, and the 8 keeps the 7 in the second place.

27. It is important to remember that each digit has two values, an *absolute* and a *relative*. The absolute value is the number of units it expresses, whatever these units may be, and is unchangeable; thus 6 always means six; sometimes, indeed, six tens; at other times six hundreds, &c. The relative value depends on the order of units indicated, and on the nature of the "simple unit."^{*}

* What has been said on this very important subject is intended principally for the teacher, though an ordinary amount of industry and intelligence will be quite sufficient for the purpose of explaining it, even to a child, particularly if each point is illustrated by an appropriate example; the pupil may be made, for instance, to arrange a number of pebbles in groups, sometimes of one, sometimes of another, and sometimes of several orders, and then be desired to express them by characters—the "unit of comparison" being occasionally changed from individuals, suppose to tens, or hundreds, or to scores, or dozens, &c. Indeed the pupils *must* be well acquainted with these introductory matters, otherwise they will contract the habit of answering without any very definite ideas of many things they may be called upon to explain, and which they should be expected *perfectly* to understand. Any trouble bestowed by the teacher at this period will be well repaid by the ease and rapidity with which the learner will afterwards advance. To be assured of this, he has only to recollect that most of his future reasonings will be derived from, and his explanations grounded on the very principles we have endeavoured to unfold. It may be taken as a truth, that what a child learns without understanding, he will acquire with disgust, and will soon cease to remember; for it is with children as with persons of more advanced years—when we appeal successfully to their understandings, the pride and pleasure they feel in the attainment of knowledge, cause the labour and the weariness which it costs to be undervalued or forgotten.

Pebbles will answer well for examples—indeed, their use in computing

ROMAN SYSTEM OF NOTATION.

28. Our ordinary numerical characters have not been always, or everywhere, used to express numbers; the letters of the alphabet naturally presented themselves for the purpose, as being already familiar, and, accordingly, were very generally adopted—for example, by the Hebrews, Greeks, Romans, &c., each, of course, using their own alphabet. The pupil should be acquainted with the Roman notation on account of its beautiful simplicity, and its being still employed in inscriptions, &c. : it is found in the following table :—

Characters. Numbers Expressed.

| | | | |
|--------------------|------------|--------|------------|
| | I. | . | One. |
| | II. | . | Two. |
| | III. | . | Three. |
| Anticipated change | IIII. | or IV. | Four. |
| Change | V. | . | Five. |
| | VI. | . | Six. |
| | VII. | . | Seven. |
| | VIII. | . | Eight. |
| Anticipated change | IX. | . | Nine. |
| Change | X. | . | Ten. |
| | XI. | . | Eleven. |
| | XII. | . | Twelve. |
| | XIII. | . | Thirteen. |
| | XIV. | . | Fourteen. |
| | XV. | . | Fifteen. |
| | XVI. | . | Sixteen. |
| | XVII. | . | Seventeen, |
| | XVIII. | . | Eighteen. |
| | XIX. | . | Nineteen. |
| | XX. | . | Twenty. |
| | XXX., &c., | . | Thirty. |

has given rise to the term *calculation*, “*calculus*” being, in Latin, a pebble; but while the teacher illustrates what he says by groups of particular objects, he must take care to notice that his remarks would be equally true of any others. He must also point out the difference between a group and its equivalent unit, which, from their perfect equality, are generally confounded. Thus he may show that a penny, while *equal* to, is not *identical* with four farthings. This seemingly unimportant remark will be better appreciated hereafter; at the same time, without inaccuracy of result, we may, if we please, consider any group *either* as a unit of the order to which it belongs, or so many of the next lower as are equivalent.

Characters. Numbers Expressed.

| | | | |
|--------------------|--|-------|---------------------------|
| Anticipated change | XL. | . . . | Forty. |
| Change | L. | . . . | Fifty. |
| | LX., &c., | . . . | Sixty, &c. |
| Anticipated change | XC. | . . . | Ninety. |
| Change | C. | . . . | One hundred. |
| | CC., &c., | . . . | Two hundred, &c. |
| Anticipated change | CD. | . . . | Four hundred. |
| Change | D. or I \bar{O} . | . . . | Five hundred, &c. |
| Anticipated change | CM. | . . . | Nine hundred. |
| Change | M. or CI \bar{O} . | . . . | One thousand, &c. |
| | \bar{V} . or I \bar{O} \bar{O} . | . . . | Five thousand. |
| | \bar{X} . or CCI \bar{O} \bar{O} . | . . . | Ten thousand, &c. |
| | I \bar{O} \bar{O} \bar{O} . | . . . | Fifty thousand, &c., |
| | CCCI \bar{O} \bar{O} \bar{O} . | . . . | One hundred thousand, &c. |

29. Thus we find that the Romans used very few characters—fewer, indeed than we do, although our system is still more simple and effective from our applying the principle of “position,” unknown to them.

They expressed all numbers by the following symbols, or combinations of them: I. V. X. L. C. D. or I \bar{O} . M., or CI \bar{O} . In constructing their system, they evidently had a quinary in view; that is, as we have said, one in which five would be the *common ratio*; for we find that they changed their character, not only at ten, ten times ten, &c.; but also at five, ten times five, &c. A purely decimal system would suggest a change only at ten, ten times ten, &c.; a purely quinary, only at five, five times five, &c. As far as notation was concerned, what they adopted was neither a decimal nor a quinary system, nor even a combination of both; they appear to have supposed *two* primary groups, one of five, the other of ten “units of comparison”; and to have formed all the other groups from these, by using ten as the *common ratio* of each resulting series.

30. They anticipated a change of character,—one unit before it would naturally occur; that is, not one “simple unit,” but one of the units under consideration. In this point of view, four is one unit before five; forty, one unit before fifty—tens being now the units under consideration; four hundred, one unit before five hundred—hundreds having become the units contemplated.

31. From the table (28) it will be seen that as often as any letter is repeated, so many times is its value repeated. Thus I, standing alone, denotes *one*, II denotes *two*, &c. So X denotes *ten*, XX *twenty*, &c.

When a letter of *less* value is placed *before* a letter of *greater* value, it *takes away* its own value from the greater; but when placed *after* it, it *adds* its own value to the greater. Thus V denotes *five*, IV denotes *four*, and VI *six*; so X denotes *ten*, IX *nine*, and XI *eleven*, &c.

A line or bar placed over any letter increases its value a *thousand-fold*. Thus V denotes *five*, \bar{V} denotes *five thousand*; X denotes *ten*, \bar{X} denotes *ten thousand*, &c.

32. To express a number by the Roman method of notation :

RULE.—Find the highest number within the given one, that is expressed by a single character, or the “anticipation” of one (28); set down that character, or anticipation, as the case may be, and take its value from the given number. Find what highest number less than the remainder is expressed by a single character, or “anticipation”; put that character or “anticipation” to the right hand of what is already written, and take its value from the last remainder: proceed thus until nothing is left.

EXAMPLE.—Set down the number eighteen hundred and forty-four, in Roman characters. One thousand expressed by M. is the highest number within the given one, indicated by one character or by an anticipation; we put down

M,

and take one thousand from the given number, which leaves eight hundred and forty-four. Five hundred, D, is the highest number within the last remainder (eight hundred and forty-four) expressed by one character, or an “anticipation”; we set down D to the right hand of M,

MD,

and take its value from eight hundred and forty-four, which leaves three hundred and forty-four. In this the highest number expressed by a single character, or an “anticipation,” is one hundred, indicated by C; which we set down, and for the same reason two other C's,

MDCCC.

This leaves only forty-four, the highest number within which, expressed by a single character or an “anticipation,” is forty, XL—an anticipation: we set this down also,

MDCCCXL.

Four, expressed by IV, still remains; which, being also added, the whole is as follows:—

MDCCCXLIV.

EXERCISE.

33. Express the following numbers in the Roman notation:—

1. Twenty-five.
2. Forty-three.
3. Sixty-seven.
4. Eighty-nine.
5. Ninety-eight.
6. One hundred and thirty-seven.
7. Three hundred and seventy-one.
8. Four hundred and two.
9. Six hundred and seventeen.
10. Nine hundred and ninety-nine.
11. One thousand four hundred and forty-six.
12. Three thousand eight hundred and five.
13. Eight thousand six hundred and seventy.
14. Twelve thousand one hundred and sixty-nine.
15. Four hundred and ninety-seven thousand, six hundred and eighty-two.

Answers.

- | | | |
|--|--|---------------|
| 1. XXV. | 2. XLIII. | 3. LXVII. |
| 4. LXXXIX. | 5. XCVIII. | 6. CXXXVII. |
| 7. CCCLXXI | 8. CDIL. | 9. DCXVII. |
| 10. CMXCIX. | 11. MCDXLVI. | 12. MMMCCCIV. |
| 13. $\overline{\text{V}}\text{MMMDCCLXX.}$ | 14. $\overline{\text{X}}\text{MMCLXIX.}$ | |
| | 15. $\overline{\text{CDXC}}\overline{\text{V}}\text{MMDCCLXXXII.}$ | |

EXERCISE.

34. Read the following expressions:—

- | | | |
|--|---|---|
| 1. XCVII | 2. CCLXXII. | 3. DCLXVIII. |
| 4. CMIX | 5. $\overline{\text{XV.}}$ | 6. $\overline{\text{V}}\text{MMMXXXIII.}$ |
| 7. $\overline{\text{XV}}\text{DCCLXXXVIII.}$ | 8. $\overline{\text{DCXLV}}\text{MCMIV.}$ | 9. $\overline{\text{XXV}}\text{XXV.}$ |
1. Ninety-seven.
 2. Two hundred and seventy-two.
 3. Six hundred and sixty-eight.
 4. Nine hundred and nine.
 5. Fifteen thousand.
 6. Eight thousand and thirty-three.
 7. Fifteen thousand eight hundred and eighty-eight.
 8. Six hundred and forty-six thousand nine hundred and four.
 9. Twenty-five thousand and twenty-five.

ARABIC SYSTEM OF NOTATION.

35. In the Common or Arabic system of Notation the same character may have different values, according to the *place* it holds with reference to the *decimal point* (25), or perhaps more strictly to the simple units. This is the principle of *position*.

36. The places occupied by the units of the different orders (23), may be described as follows:—simple units, one place to the left of the decimal point, expressed, or understood; tens, two places; hundreds, three places, &c.

37. When, therefore, we are desired to write any number, we have merely to put down the digits expressing the amounts of the different units in their proper places, according to the order to which each belongs. If, in the given number, there is any “place” in which there is no digit, a cypher must be set down in that place, when required to keep another digit in its own *position*.—But a cypher produces no effect, when it is not between one or more digits and the decimal point; thus, 0536, 536.0, and 536 would mean the same thing—the first is, however, incorrect. 536 and 5360 are different; in the latter case the cypher affects the value, because it alters the *position* of the digits.

EXAMPLE.—Let it be required to set down six hundred and two. The six must be in the third, and the two in the first place; for this purpose we are to put a cypher between the 6 and 2—thus 602. Without a cypher the six would be in the second place—thus, 62; and would mean, not six hundreds, but six tens.

38. In numerating, we begin with the digits of the highest order, and proceed downwards, stating the number which belongs to each order.

To facilitate notation and numeration, it is usual to divide the places occupied by the different orders of units into periods. For a certain distance, the English and French methods of division agree; the English billion is, however, a thousand times greater than the French. This discrepancy is not of much importance, since we are rarely obliged to use so high a number;—we shall prefer the French method. To give some idea of the amount of a billion, it is only necessary to remark, that, according to the English method of notation, there has not been one billion of seconds since the birth of Christ. Indeed, to reckon even a million, counting on an average three per second for eight hours a day, would require nearly 12 days. The following are the two methods:

| ENGLISH METHOD. | FRENCH METHOD. |
|---------------------------------------|-------------------------------|
| Hunds. of Thous. of Quadrillions..... | Hundreds of Octillions..... |
| Tens of Thous. of Quadrillions | Tens of Octillions..... |
| Hundreds of Quadrillions | Hundreds of Septillions..... |
| Tens of Quadrillions | Tens of Septillions..... |
| Quadrillions | Septillions..... |
| Hunds. of Thous. of Trillions..... | Hundreds of Sextillions..... |
| Thousands of Trillions | Tens of Sextillions..... |
| Hundreds of Trillions..... | Sextillions..... |
| Trillions | Hundreds of Quintillions..... |
| Hunds. of Thous. of Billions..... | Quintillions..... |
| Tens of Thous. of Billions | Hundreds of Quadrillions..... |
| Thousands of Billions | Tens of Quadrillions..... |
| Hundreds of Billions | Quadrillions..... |
| Tens of Billions | Hundreds of Trillions..... |
| Billions | Tens of Trillions..... |
| Hunds. of Thous. of Millions..... | Trillions..... |
| Tens of Thousands of Millions | Hundreds of Billions..... |
| Thousands of Millions | Tens of Billions..... |
| Hundreds of Millions | Billions..... |
| Tens of Millions | Hundreds of Millions..... |
| Millions..... | Tens of Millions..... |
| Hundreds of Thousands..... | Millions..... |
| Tens of Thousands..... | Hundreds of Thousands..... |
| Thousands..... | Tens of Thousands..... |
| Hundreds..... | Thousands..... |
| Tens..... | Hundreds..... |
| Units..... | Tens..... |
| | Units..... |

39. *Use of Periods.*—For the purpose of reading or writing numbers, we divide them by separating points, into periods—the first separating point being the decimal point, expressed or understood, and the other separating points being placed after every third digit, or place, to the right and left of the decimal point. Each period has three places—of which one or more may be occupied by digits. The lowest place in every period—or that to the *right* hand, is the “units’” place of that period: and the highest, the “hundreds’” place. And this is true, whether the period is to the left or to the right of the decimal point.

40. The period to the left of the decimal point contains the simple units. The first period to the left of the *units’* period, contains the *thousands*; and the first period to the right of it, the *thousandths*. The second period to the left of the *units’* period, contains the *millions*; and the second to the right of it, the *millionths*. The third period to the left of the *units’* period, contains the *billions*; and the third to the right of it, the *billionths*. The fourth period to the left of the *units’* period, contains the *trillions*; and the fourth to the right of it, the *trillionths*. The fifth period to the left of the *units’* period, con-

tains the *quadrillions*; and the fifth to the right of it, the *quadrillionths*. The sixth period to the left of the units' period, contains the *quintillions*; and the sixth to the right of it, the *quintillionths*. The seventh period to the left of the units' period, contains the *sextillions*; and the seventh to the right of it, the *sextillionths*. The eighth period to the left of the units' period, contains the *septillions*; and the eighth to the right of it, the *septillionths*. The ninth period to the left of the units' period, contains the *octillions*; and the ninth to the right of it, the *octillionths*. The tenth period to the left of the units' period, contains the *nonillions*; and the tenth to the right of it, the *nonillionths*.

The pupil should be made perfectly familiar with the names of the *periods* and of the *places* in each period—so as to be able, without the slightest hesitation, to name the period and place to which any digit belongs, or into which it ought to be put. When he can read or write any *one* digit, belonging to any period and place, he should be taught to read and write a number consisting of *two, three, four, &c.* digits, whether they are close together, or separated by any number of ciphers.

The whole of what has been said above will become more evident from an attentive consideration of the following table :

| | | | | | | | | | | | |
|------------------|---------------|--------------|--------------|---------------|-----------|-----------------|----------------|----------------|-----------------|--------------------|--------------------|
| of Quadrillions. | of Trillions. | of Billions. | of Millions. | of Thousands. | of Units. | of Thousandths. | of Millionths. | of Billionths. | of Trillionths. | of Quadrillionths. | of Quintillionths. |
| Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds | Hundreds |
| Tens | Tens | Tens | Tens | Tens | Tens | Tens | Tens | Tens | Tens | Tens | Tens |
| Units | Units | Units | Units | Units | Units | Units | Units | Units | Units | Units | Units |
| 2 | 0 | 1 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 3 | 1 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 |
| 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 |
| 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 9 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 |
| 0 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 1 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 2 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 |
| 3 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 |
| 4 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 5 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 |
| 6 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 7 | 9 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 8 | 0 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 9 | 1 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 0 | 2 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 |
| 1 | 3 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 |
| 2 | 4 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 |
| 3 | 5 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 4 | 6 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 5 | 7 | 9 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 6 | 8 | 0 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 |
| 7 | 9 | 1 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 8 | 0 | 2 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 9 | 1 | 3 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 |
| 0 | 2 | 4 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 |
| 1 | 3 | 5 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 2 | 4 | 6 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 3 | 5 | 7 | 9 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 4 | 6 | 8 | 0 | 2 | 8 | 3 | 7 | 4 | 5 | 6 | 2 |
| 5 | 7 | 9 | 1 | 3 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 6 | 8 | 0 | 2 | 4 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 7 | 9 | 1 | 3 | 5 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 8 | 0 | 2 | 4 | 6 | 2 | 8 | 3 | 7 | 4 | 5 | 6 |
| 9 | 1 | 3 | 5 | 7 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 0 | 2 | 4 | 6 | 8 | 3 | 7 | 4 | 5 | 6 | 2 | 8 |
| 1 | 3 | 5 | 7 | 9 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 2 | 4 | 6 | 8 | 0 | 2 | 8 | 3 | 7 | 4 | 5 | 6 |
| 3 | 5 | 7 | 9 | 1 | 3 | 7 | 4 | 5 | 6 | 2 | 8 |
| 4 | 6 | 8 | 0 | 2 | 4 | 5 | 6 | 2 | 8 | 3 | 7 |
| 5 | 7 | 9 | 1 | 3 | 5 | 6 | 2 | 8 | 3 | 7 | 4 |
| 6 | 8 | 0 | 2 | 4 | 6 | 2 | 8 | 3 | 7 | 4 | 5 |
| 7 | 9 | 1 | 3 | 5 | 7 | 4 | 5 | 6 | 2 | 8 | 3 |
| 8 | 0 | 2 | 4 | 6 | 8 | 3 | 7 | 4 | 5 | 6 | 2 |
| 9 | 1 | 3 | 5 | 7 | 9 | 7 | 4 | 5 | 6 | 2 | 8 |
| 0 | 2 | 4 | 6 | 8 | 0 | 2 | 8 | 3 | 7 | 4 | 5 |
| 1 | 3 | 5 | 7 | 9 | 1 | 3 | 7 | 4 | 5 | 6 | 2 |
| 2 | 4 | 6 | 8 | 0 | 2 | 4 | 5 | 6 | 2 | 8 | 3 |
| 3 | 5 | 7 | 9 | 1 | 3 | 5 | 6 | 2 | 8 | 3 | 7 |
| 4 | 6 | 8 | 0 | 2 | 4 | 6 | 2 | 8 | 3 | 7 | 4 |
| 5 | 7 | 9 | 1 | 3 | 5 | 7 | 4 | 5 | 6 | 2 | 8 |
| 6 | 8 | 0 | 2 | 4 | 6 | 8 | 3 | 7 | 4 | 5 | 6 |
| 7 | 9 | 1 | 3 | 5 | 7 | 9 | 7 | 4 | 5 | 6 | 2 |
| 8 | 0 | 2 | 4 | 6 | 8 | 0 | 2 | 8 | 3 | 7 | 4 |
| 9 | 1 | 3 | 5 | 7 | 9 | 1 | 3 | 7 | 4 | 5 | 6 |
| 0 | 2 | 4 | 6 | 8 | 0 | 2 | 4 | 5 | 6 | 2 | 8 |
| 1 | 3 | 5 | 7 | 9 | 1 | 3 | 5 | 6 | 2 | 8 | 3 |
| 2 | 4 | 6 | 8 | 0 | 2 | 4 | 6 | 2 | 8 | 3 | 7 |
| 3 | 5 | 7 | 9 | 1 | 3 | 5 | 7 | 4 | 5 | 6 | 2 |
| 4 | 6 | 8 | 0 | 2 | 4 | 6 | 8 | 3 | 7 | 4 | 5 |
| 5 | 7 | 9 | 1 | 3 | 5 | 7 | 9 | 7 | 4 | 5 | 6 |
| 6 | 8 | 0 | 2 | 4 | 6 | 8 | 0 | 2 | 8 | 3 | 7 |
| 7 | 9 | 1 | 3 | 5 | 7 | 9 | 1 | 3 | 7 | 4 | 5 |
| 8 | 0 | 2 | 4 | 6 | 8 | 0 | 2 | 4 | 5 | 6 | 2 |
| 9 | 1 | 3 | 5 | 7 | 9 | 1 | 3 | 5 | 6 | 2 | 8 |

EXAMPLES.—Let it be required to read off the following number, 576934* We put a point to the left of the 9, and find that there are *exactly* two periods—thus, 576,934; this does not always occur, as the highest or lowest period is often imperfect, consisting only of one or two digits. Dividing the number thus into parts, shows at once that 5 is in the third place of the second period—that is, in the *Hundreds'* place of the *Thousands'* period; and therefore, that it expresses five hundred thousands; that the 7, being in the second place of the same period indicates tens of thousands; and the 6, being in the first, indicates thousands. The 9, being in the

third place of the first period, indicates hundreds of units; the 3, being in the second place of the same period, indicates tens of units; and the 4, being in the first, indicates units ("of comparison," or "simple units"). The number, therefore, may be read as follows—"five hundreds of thousands, seven tens of thousands, and six thousands; nine hundreds of units, three tens of units, and four units"; or more briefly, "five hundred and seventy-six thousand nine hundred and thirty-four."

41. To prevent the separating point or that which divides into periods, from being mistaken for the decimal point, the former should be a comma (,)—the latter a full stop (.) Without this distinction, two numbers which are very different might be confounded: thus, 498^s763, and 498,763, one of which is a thousand times greater than the other. After awhile we may dispense with the separating point, though it is convenient to retain it with large numbers, as they are then read with greater ease.

42. To write down any integral or whole number, it is merely necessary to remember the order of the periods, and that every period contains three places, each of which must be filled, either by a digit or a cipher. The one, two, or three digits, belonging to the highest period are first written in their appropriate places; then the next lower period is filled with the digits, or ciphers belonging to it; afterwards the next; and so on, till the whole number is set down.

EXAMPLE.—Let it be required to write the number seventy-three trillions two hundred and nine billions eighteen thousand and six. The highest period here mentioned is that of trillions, which we know to be the fifth to the left of the decimal point (40). We therefore set down the digits 73, bearing in mind that we are to put in four complete periods, or twelve places between the 3 and the decimal point. The next period we have is that of billions, which we fill with digits 209 (two hundred and nine). The next period, that of millions, has no significant figures, and we accordingly fill it thus, 000. We now come to the period of thousands, in which we have the digits 18, but, inasmuch as the third place of this period must also be filled, we insert there a cipher, and the full period becomes 018. Lastly, the lowest period, or that of units, is to contain only the digit 6,—the other two places being filled with ciphers, the complete period is written 006. Now setting these periods one after the other in their proper order, we obtain for the entire number the expression, 73,209,000,018,006.

43. To write down any decimal number we proceed very much in the same way. We have to remark, that in any decimal the last digit to the right gives the denomination to the number. Thus, .68 is read sixty-eight hundredths; .4078 is read four thousand and seventy-eight tenths of thousandths, &c.

Now, when we wish to write any decimal, we first ascertain how many places the proposed denomination or order is to the right of the decimal point; and then, if the given digits will not bring the number to its proper position, we insert between these digits and the decimal point the requisite number of ciphers.

EXAMPLE. 1.—Let it be required to write the number, seven hundred and sixteen thousand and eighty-nine billionths. Now we know (40) that billionths occupy the 9th place to the right of the decimal point. Were we to place the decimal point immediately before the digits themselves, thus, .716089, they would express not so many billionths but so many millionths; since millionths occupy the 6th and billionths the 9th place. It is obvious, then, that to give the digits their proper value, we must insert three ciphers between them and the decimal point, and the number is then correctly written .000,716,089.

EXAMPLE 2.—Write the number six thousand two hundred and one hundredths of trillionths. From (40) we know that hundredths of trillionths occupy the 14th place. The given digits (6201) being only four in number, require the aid of ten ciphers in order to fill the 14 places, and the number is, thus written, '000,000,000,062,01.

EXAMPLE 3.—Write the number, six millions seven hundred and twenty-seven thousand and twelve tenths of billionths. The given digits, 6727012, are only seven in number, while the denomination *tenths of billionths* implies that ten places must be filled. We have, therefore, to insert three ciphers between the given digits and the decimal point, and the resulting expression, '000,672,701,2, represents the given number.

44. The simple units are, as we have said, always found in the first period to the left of the decimal point. The digits to the left hand, progressively increase in a tenfold degree—those occupying the first place to the left of the simple units being ten times greater than the simple units; those occupying the second place, ten times greater than those which occupy the first, and one hundred times greater than the units of comparison themselves; and so on. Moving a digit one place to the *left*, multiplies it by ten—that is, makes it ten times greater; moving it two places, multiplies it by one hundred—that is, makes it one hundred times greater; and so of the rest. If all the digits of a quantity be moved one, two, &c., places to the left, the whole is increased ten, one hundred, &c., times—as the case may be. On the other hand, moving a digit, or a quantity one place to the *right*, divides it by ten, that is makes it ten times smaller than before; moving it two places, divides it by one hundred, or makes it one hundred times smaller, &c.

45. We possess this power of easily increasing, or diminishing any number in a tenfold, &c., degree, whether the digits are all at the right, or all at the left of the decimal point; or partly at the right or partly at the left. And the pupil must remember that the quantities increase in a tenfold degree to the left, and decrease in the same degree to the right wherever the decimal point may happen to be. We therefore put quantities ten times less than simple units one place to the right of them, just as we put those which are ten times less than hundreds, &c., one place to the right of hundreds, &c. Quantities to the left of the decimal point are called *integers*, because none of them is less than a whole "simple unit"; and those to the right of it, *decimals*. When there are decimals in a given number, the decimal point is *always* expressed, and is found at the right-hand side of the simple units.

46. The periods to the left of the decimal point may be called the *ascending*, and those to the right of it the *descending* series:—taken together, however, they constitute but *one* series, which is an ascending or a descending series, according as it is read from right to left or from left to right. Periods that are equally distant from the units of comparison bear a very close relation to

each other, which is indicated even by the similarity of their names; the only difference being in the terminations (40). We have seen also, that when we divide integers into periods (40), the first separating point must be put to the right of the thousands. In dividing decimals into periods, the first point must be put to the right of the thousandths also.

47. Care must be taken not to confound what we now call "decimals," with what we shall hereafter designate "decimal fractions"; for they express equal, but not identically the same quantities—the decimals being what shall be termed the "quotients" of the corresponding decimal fractions. This remark is made here to anticipate any inaccurate idea on the subject, in those who already know something of arithmetic.

48. There is no reason for treating integers and decimals by different rules, and at different times, since they follow precisely the same laws, and constitute parts of the very same series of numbers (46). Besides, any quantity may, as far as the decimal point is concerned, be expressed in different ways; for this purpose we have merely to change the unit of comparison. Thus, let it be required to set down a number indicating five hundred and seventy-four men. If the unit be *one man*, the quantity would stand as follows, 574. If a band of *ten men*, it would become 57·4—for as each man would then constitute only the tenth part of the "unit of comparison," four men would be only four tenths, or 0·4; and, since ten men would form but one unit, seventy men would be merely seven simple units, or 7, &c. Again, if it were a band of one *hundred men*, the number must be written 5·74; and lastly, if a band of a *thousand men*, it would be 0·574. Should the "unit" be a band of a dozen, or a score of men, the change would be still more complicated; as, not only the position of a decimal point, but the very digits also, would be altered.

49. It is not necessary to remark that moving the decimal point so many places to the *left*, or the digits an equal number of places to the *right*, amounts to the same thing.

Sometimes in changing the decimal point, one or more ciphers are to be added; thus, when we move 42·6 three places to the left, it becomes 42600; when we move 27 five places to the right it is 0·00027, &c.

50. It follows from what we have said, that a decimal, though less than what constitutes the unit of comparison, may itself consist of not only one, but several individuals. Of course it will often be necessary to indicate the nature of the "simple units"; as 3 scores, 5 dozen, 6 men, 7 companies, 8 regiments, &c. But its nature does not affect the abstract properties of numbers; for it is true to say that seven and five, when added

together, make twelve, whatever the unit of comparison may be :—provided, however, that the *same* standard be applied to both ; thus 7 men and 5 men are 12 men ; but 7 men and 5 horses are neither 12 men nor 12 horses ; 7 men and 5 dozen men are neither 12 men nor 12 dozen men. When, therefore, numbers are to be compared, &c., they must have the same unit of comparison :—or, without altering their value, they must be reduced to those which have. Thus we may consider 5 *tens* of men to become 50 *individual* men—the unit being altered from *ten men* to *one man*, without the value of the quantity being changed. This principle must be kept in mind from the very commencement, but its utility will become more obvious hereafter.

EXERCISE.

51. Write down the following Numbers :—

1. One hundred and ninety-four.
2. One thousand and seventy-six.
3. Twenty thousand five hundred and eight.
4. Two hundred and one thousand and three.
5. Eighty millions four thousand and thirty-three.
6. Sixteen quadrillions five hundred and ninety-seven trillions three billions forty-four millions and ninety-one.
7. Ninety-seven hundredths.
8. Six hundred and forty-three thousandths.
9. One hundred and twenty-two thousand and eighty-nine millionths.
10. Thirty-nine tenths of millionths.
11. Sixty-three hundredths of trillionths.
12. Seventeen billions four thousand and one, and nine hundred and sixty-seven billionths.
13. Seven trillions eight hundred and two billions twenty-three thousand and eleven, and nine thousand nine hundred and ninety-nine billionths.
14. One quadrillion one trillion one billion one million one thousand one hundred and one, and one trillionth.
15. Eight hundred and ninety-six trillions and two, and nine hundred and four hundredths of millionths.

Answers.

- | | | |
|------------|-----------------------------------|-----------------------|
| 1. 194. | 2. 1076. | 3. 20508. |
| 4. 201003. | 5. 80004033. | 6. 16597003044000091. |
| 7. 97. | 8. 643. | 9. 122089. |
| | 10. 0000039. | |
| | 11. 000000000000063. | |
| | 12. 17000004001000000967. | |
| | 13. 7802000023011000009999. | |
| | 14. 1001001001001101000000000001. | |
| | 15. 89600000000000200000904. | |

EXERCISE.

52. Read the following numbers :—

- | | |
|--------------------|---------------------------|
| 1. 904. | 7. 604·03. |
| 2. 7060. | 8. 90767·004003. |
| 3. 90004. | 9. 9001·00070306. |
| 4. 40300201. | 10. 1237·9134671342913. |
| 5. 7060504030. | 11. ·00100100100101. |
| 6. 70003000000400. | 12. 100·2003004005006007. |

Answers.

- Nine hundred and four.
- Seven thousand and sixty.
- Ninety thousand and four.
- Forty millions three hundred thousand two hundred and one.
- Seven billions sixty millions five hundred and four thousand and thirty.
- Seventy trillions three billions and four hundred.
- Six hundred and four, and three hundredths.
- Ninety thousand seven hundred and sixty-seven, and four thousand and three millionths.
- Nine thousand and one, and seventy thousand three hundred and six hundredths of millionths.
- One thousand two hundred and thirty seven, and nine trillions one hundred and thirty-four billions six hundred and seventy-one millions three hundred and forty-two thousand nine hundred and thirteen tenths of trillionths.
- One hundred billions one hundred millions one hundred thousand one hundred and one hundredths of trillionths.
- One hundred, and two quadrillions three trillions four billions five millions six thousand and seven tenths of quadrillionths.

ON THE DENOMINATION OF NUMBERS.

53. When two numbers have the same unit they are said to be of the same denomination ; when the units are not the same, they are said to be of different denominations. For example, 16 shillings and 28 shillings are two numbers of the same denomination ; but 23 shillings and three farthings are not of the same denomination, the unit of 23 shillings being one *shilling*, and of three farthings, one *farthing*. *The kind of unit always expresses the denomination.*

Even in abstract or simple numbers, different names are given to the units as we proceed to the right or left of the decimal point, viz., simple units, or units of the first order; tens, or units of the second order; hundreds, or units of the third order, &c. Considered in this relation to each other, these units may be regarded as denominate numbers.

The following Tables show the various kinds of denominate numbers in general use, and also the relative values of their different units.

TABLES OF MONEY, WEIGHTS, AND MEASURES.

STERLING MONEY.

54. The denominations are pounds, shillings, pence, and farthings.

TABLE.

| | | | | |
|-------------------|------|-------------|--------|-----------|
| 4 farthings (qr.) | make | 1 penny | marked | <i>d.</i> |
| 12 pence | " | 1 shilling, | " | <i>s.</i> |
| 20 shillings | " | 1 pound, | " | £. |

| | | | | |
|------------|---|-----------|-----------|---------|
| <i>qr.</i> | = | <i>d.</i> | | |
| 4 | = | 1 | <i>s.</i> | |
| 48 | = | 12 | 1 | £ |
| 960 | = | 240 | = | 20 = 1. |

Other English coins, some of them now out of use :

| | | | | | |
|--------------|---|-----------|-------|---|---------|
| Moidore | = | 27s. | Noble | = | 6s. 8d. |
| Guinea | = | 21s. | Crown | = | 5s. |
| Pistole | = | 16s. 10d. | Angel | = | 10s. |
| Mark or Merk | = | 13s. 4d. | Groat | = | 4d. |

The letters *£ s. d.* and *qr.* are the initials of the Latin words, *libra*, *solidus*, *denarius*, and *quadrans*, which respectively signify a *pound*, a *shilling*, a *penny*, and a *farthing*, or quarter. The mark */*, which sometimes separates the shillings and pence, is a corruption of the long *f* (*s*), arising from the rapidity with which it is made.

It is now customary to write farthings as fractions of a penny, as $\frac{1}{4}$ d. $\frac{2}{4}$ d. $\frac{3}{4}$ d., to represent 1 qr., 2 qr., and 3 qr.

Sterling money is supposed to have received its name from the *Esterlings* or German traders in England, by whom it is said to have been first coined. The pound is so called, because in ancient times it was equal to a pound Troy of silver. Its present value in Canada is \$4.8666, and hence the value of an English shilling is 24¢ cents. The guinea was so called from being originally coined from gold brought from Guinea, on the coast of Africa,

The present standard gold coin of Great Britain consists of 22 parts *pure gold* and 2 parts of copper. The standard silver coin consists of 37 parts *pure silver* and 3 parts copper. In copper coin 24 pence weigh a pound avoirdupois.

FEDERAL MONEY.

55. *Federal money* is the currency of the United States. The denominations are eagles, dollars, dimes, cents, and mills.

TABLE.

| | | | | |
|---------------|--------|-----------|--------|-----|
| 10 mills (m.) | make | 1 cent, | marked | ct. |
| 10 cents | " | 1 dime, | " | d. |
| 10 dimes | " | 1 dollar, | " | \$. |
| 10 dollars | " | 1 eagle, | " | E. |
| m. | | ct. | | |
| 10 = | | 1 | d. | |
| 100 = | 10 = | 1 | \$. | |
| 1000 = | 100 = | 10 | 1 | E. |
| 10000 = | 1000 = | 100 | 10 | 1. |

The sign \$ is the symbol for the old Spanish coin of 8 reals. On one side of the Spanish real the pillars of Hercules were represented supporting the world—on the piece of eight reals the pillars were retained and the 8 written over them—thus \$. Many however consider the sign \$ a contraction of the letters U. S., the initials of *United States*, made by dropping the curve of the U and writing the S over it.

The present standard for both *gold* and *silver* coin in the United States is 900 parts of pure metal and 100 parts of alloy. The alloy for gold is silver and copper; that for silver is pure copper.

The gold coins are the Eagle, the Double Eagle, Half Eagle, Quarter Eagle, and Dollar; the silver coins are the Dollar, Half Dollar, Quarter Dollar, Dime, and Half Dime; the copper coins are the Cent and Half Cent; Mills are never coined.

OLD CANADIAN MONEY.

56. The denominations are pounds, dollars, shillings, pence, and farthings.

TABLE.

| | | | | |
|-------------|-------|-------------|--------|-----|
| 4 farthings | make | 1 penny, | marked | d. |
| 12 pence | " | 1 shilling, | " | s. |
| 5 shillings | " | 1 dollar, | " | \$. |
| 4 dollars | " | 1 pound, | " | £. |
| qr. | | d. | | |
| 4 = | | 1 | s. | |
| 48 = | 12 = | 1 | \$. | |
| 240 = | 60 = | 5 = | 1 | £. |
| 960 = | 240 = | 20 = | 4 = | 1. |

NOTE.—Every 3d. of the old coinage is equal to 5 cents of the new. The York shilling was equal to the eighth part of a \$, or to 7½d. or to 12½ cents.

NEW CANADIAN OR DECIMAL MONEY.

57. The denominations are dollars and cents.

The coins are cents, five-cent pieces, ten-cent pieces, and twenty-cent pieces.

100 cents (c) make 1 dollar, marked \$.

AVOIRDUPOIS WEIGHT

58. Is used in weighing heavy articles. Its name is derived from French—and ultimately from Latin words signifying “to have weight.” Its denominations are tons, hundredweights, quarters, pounds, ounces, and drams.

TABLE.

| | | | | |
|------------|---------|------------------|--------|---------|
| 16 drams | make | 1 ounce, | marked | oz. |
| 16 ounces | “ | 1 pound, | “ | lb. |
| 25 pounds | “ | 1 quarter, | “ | qr. |
| 4 quarters | “ | 1 hundredweight, | “ | cwt. |
| 20 cwt. | “ | 1 ton, | “ | t. |
| d. | oz. | | | |
| 16 = | 1 | lb. | | |
| 256 = | 16 = | 1 | qr. | |
| 6400 = | 400 = | 25 = | 1 | cwt. |
| 25600 = | 1600 = | 100 = | 4 = | 1 t. |
| 512000 = | 32000 = | 2000 = | 80 = | 20 = 1. |

It was formerly the custom to allow 28 lbs. to the quarter, 112 lbs. to the hundredweight, and 2240 to the ton. This has now fallen into disuse; and among merchants in Canada the qr., cwt. and ton are universally considered as respectively equal to 25 lbs., 100 lbs., and 2000 lbs. The Custom Houses continue to regard the cwt. as equal to 112 lbs., and some few articles are still weighed by the old cwt. by farmers and others.

TROY WEIGHT.

59. The denominations of Troy Weight are pounds, ounces, pennyweights, and grains.

TABLE.

| | | | | |
|------------------|------|----------------|--------|------|
| 24 grains (grs.) | make | 1 pennyweight, | marked | dwt. |
| 20 pennyweights | “ | 1 ounce, | “ | oz. |
| 12 ounces | “ | 1 pound, | “ | lb. |

| | | | |
|--------|-------|------|-----|
| grs. | dw. | oz. | lb. |
| 24 = | 1 | | |
| 480 = | 20 = | 1 | |
| 5760 = | 240 = | 12 = | 1. |

This weight was introduced into Europe from Cairo, in Egypt, and was first adopted in Troyes, a city of France—whence its name. It is used in philosophy, in weighing gold, precious stones, &c.

NOTE.—The origin of all weights used in England, was a grain of wheat taken from the middle of the ear and well dried. A weight equal to 32 of these grains was called a *Pennyweight*, being equal to the weight of a silver penny then in use; 20 of these pennyweights constituted an ounce, which was the 12th part of a pound (Lat "uncia," a 12th part—compare "*inch*," the twelfth part of a foot). In later times the pennyweight came to be divided into 24 equal parts instead of 32, but these still retain the name of grains.

The "*Carat*," which is equal to about four grains (somewhat less than Troy grains), is used in weighing diamonds. The term carat is also applied in estimating the fineness of gold: the latter, when perfectly pure, is said to be "*24 carats fine*." If there are 23 parts gold, and one part some other material, the mixture is said to be "*23 carats fine*"; if 22 parts out of the 24 are gold, it is "*22 carats fine*," &c. The whole mass is, in all cases supposed to be divided into 24 parts, of which the number consisting of gold is specified. Our gold coin is 22 carats fine; pure gold, being very soft, would too soon wear out. The degree of fineness of gold articles is marked upon them at the Goldsmiths' Hall; thus we generally perceive "*18*" on the cases of gold watches: this indicates that they are "*18 carats fine*"—the lowest degree of purity which is stamped.

| | |
|----------------------------|-------|
| | grs. |
| A Troy ounce contains..... | 480 |
| An Avoirdupois ounce | 437½ |
| A Troy pound..... | 5,760 |
| An Avoirdupois pound | 7,000 |

A Troy pound is equal to 372·965 French grammes.

175 Troy pounds are equal to 144 avoirdupois; 175 Troy are equal to 192 avoirdupois ounces.

APOTHECARIES' WEIGHT.

60. The denominations of Apothecaries' Weight are pounds, ounces, drams, scruples, and grains.

TABLE.

| | | | | | |
|------------------|------|------------|--------|--------|----|
| 20 grains (grs.) | make | 1 scruple, | marked | sc. or | Ⓢ. |
| 3 scruples | " | 1 dram, | " | dr. or | ℥. |
| 8 drams | " | 1 ounce, | " | oz. or | ℥. |
| 12 ounces | " | 1 pound, | " | lb. | |

| | | | | | |
|--------|-------|------|------|----|-----|
| grs. | Ⓢ | | | | |
| 20 = | 1 | 3 | | | |
| 60 = | 3 = | 1 | ℥ | | |
| 480 = | 24 = | 8 = | 1 | | |
| 5760 = | 288 = | 96 = | 12 = | 1. | lb. |

Apothecaries mix their medicines by this weight, but buy and sell by *avoirdupois*.

The pound and ounce of this weight are the same as in Troy weight.

LONG MEASURE.

61. The denominations of Long Measure are leagues, miles, furlongs, rods, yards, feet, inches, and lines.

TABLE.

| | | | | |
|--------------------|------|--|--------|-----------|
| 12 lines (l.) | make | 1 inch, | marked | in. |
| 12 inches | " | 1 foot, | " | ft. |
| 3 feet | " | 1 yard, | " | yd. |
| 5½ yards | " | 1 rod, pole, or perch, | | rd. or p. |
| 40 rods or perches | " | 1 furlong, | " | fur. |
| 8 furlongs | " | 1 mile, | " | m. |
| 3 miles | " | 1 league, | " | lea. |
| 69⅓ miles (nearly) | " | 1 degree or 360th part of the earth's circumference. | | |

| | | | | | |
|---------|--------|--------|-------|------|----|
| in | ft. | yd. | rd. | fur. | m. |
| 12 = | 1 | | | | |
| 36 = | 3 = | 1 | | | |
| 198 = | 16½ = | 5½ = | 1 | | |
| 7920 = | 660 = | 220 = | 40 = | 1 | m. |
| 63360 = | 5280 = | 1760 = | 320 = | 8 = | 1. |

100 links, 4 rods, or 22 yards, make 1 Gunter's chain. Each link therefore is equal to $7\frac{92}{100}$ inches.

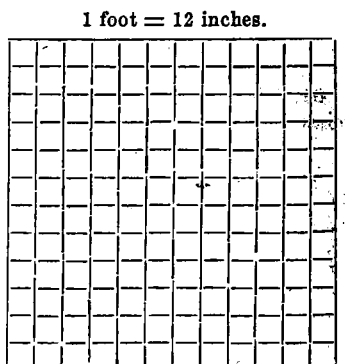
Eleven Irish are equal to 14 English miles. The Paris foot is equal to 12·792 English inches, the Roman foot to 11·604 English inches, and the French metre to 39·383 English inches.

| | | |
|-------------|------|------------------------------------|
| 4 inches | make | 1 hand (used in measuring horses). |
| 3 inches | " | 1 palm. |
| 18 inches | " | 1 cubit. |
| 3 feet | " | a common pace. |
| 5 feet | " | a Roman pace. |
| 6 feet | " | a fathom. |
| 120 fathoms | " | a cable's length. |

SQUARE MEASURE.

62. This measure is used for estimating artificers' work, such as flooring, plastering, painting, paving, &c., and, in short, any kind of work where surface alone is concerned. It is always employed in measuring land, and hence it is frequently called Land Measure.

A square is a four-sided figure having all of its sides equal and perpendicular one to another. If the length of each side be an inch, a foot, or a yard, &c., the square is called a square inch, a square foot, or a square yard, &c. It will be observed from the adjacent figure that a square foot contains 12×12 or 144 square inches, and similarly a square yard may be shown to contain 3×3 or 9 square feet.



The denominations of Square Measure are square miles, acres, roods, square perches, square yards, square feet, and square inches.

TABLE.

| | | | | |
|------------------------------|------|----------------|--------|---------|
| 144 square inches | make | 1 square foot, | marked | sq. ft. |
| 9 square feet | " | 1 square yard, | " | sq. yd. |
| $30\frac{1}{4}$ square yards | " | 1 square rod, | " | sq. rd. |
| 40 square rods | " | 1 rood, | " | r. |
| 4 roods | " | 1 acre, | " | a. |
| 640 acres | " | 1 square mile, | " | s. m. |

| | | | | |
|-----------|--------------------|-------------------|---------|---------|
| sq. in. | sq. ft. | sq. yd. | | |
| 144 = | 1 | | | |
| 1296 = | 9 = | 1 | sq. rd. | |
| 39204 = | $272\frac{1}{4}$ = | $30\frac{1}{4}$ = | 1 | r. |
| 1568160 = | 10890 = | 1210 = | 40 = | 1 acre. |
| 6272640 = | 43560 = | 4840 = | 160 = | 4 = 1. |

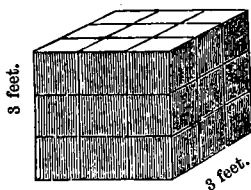
63. In measuring land, Gunter's chain is used. It is divided into 100 links.

| | | | | |
|--------------------------|------|-----------------|--------|--------|
| $7\frac{92}{100}$ inches | make | 1 link, | marked | l. |
| 100 links or 4 rods | " | 1 chain, | " | c. |
| 80 chains | " | 1 mile, | " | m. |
| 10000 square links | " | 1 square chain, | " | sq. c. |
| 10 square chains | " | 1 acre. | " | a. |

SOLID OR CUBIC MEASURE.

64. This measure is used for finding the solid contents of timber, stone, &c. A cube is a solid bounded by six equal surfaces or squares, and having eight equal edges. It is called a cubic inch, a cubic foot, or a cubic yard, according as each of these edges is an inch, a foot, or a yard in length.

The accompanying figure represents a cubic yard—each edge being 3 feet in length. The top, which is equal to the base, contains 3×3 or 9 square feet; hence, if it were only one foot in height it would contain 9 cubic feet; but it is three feet in height, and must therefore contain 9×3 or 27 cubic feet. A cubic yard then contains $3 \times 3 \times 3$ or 27 cubic feet.



Similarly it may be shown that a cubic foot contains $12 \times 12 \times 12$ or 1728 cubic inches.

The denominations of Cubic Measure are cords, tons, cubic feet, and cubic inches.

TABLE.

| | |
|---------------------------------|------------------------------------|
| 1728 cubic inches | make 1 c. ft. marked c. ft. |
| 27 cubic feet | “ 1 cubic yd., “ c. yd. |
| *40 c. ft. of round timber, or | } “ 1 ton, “ ton. |
| 50 c. ft. of sq. or hewn timber | |
| 128 cubic feet | make 1 cord of firewood, marked c. |

| | | |
|--------|--------|--------|
| c. in. | c. ft. | |
| 1728 | = 1 | c. yd. |
| 46656 | = 27 | = 1. |

A pile of cord-wood 4 feet high, 4 feet wide, and 8 feet long, contains 128 cubic feet or one cord. One foot in length of such a pile is called a *cord-foot*. It is equal to 16 solid feet, and is consequently equivalent to the eighth part of a cord.

CLOTH MEASURE.

65. The denominations of Cloth Measure are French ells, English ells, Flemish ells, quarters, nails, and inches.

* A ton of round timber is that quantity of timber which, when hewn, will make 40 cubic feet.

TABLE.

| | | | | |
|-----------------|------|----------------|--------|---------------|
| 2½ inches (in.) | make | 1 nail, | marked | na. |
| 4 nails | " | 1 quarter, | " | qr. |
| 3 quarters | " | 1 Flemish ell, | " | Fl. e. |
| 4 quarters | " | 1 yard, | " | yd. |
| 5 quarters | " | 1 English ell, | " | E. e. |
| 6 quarters | " | 1 French ell, | " | F. e. |
| in. | na. | | | |
| 2½ = | 1 | qr. | | |
| 9 = | 4 = | 1 | Fl. e. | |
| 27 = | 12 = | 3 = | 1 | yd. |
| 36 = | 16 = | 4 = | 1½ = | 1 Eng. e. |
| 45 = | 20 = | 5 = | 1½ = | 1¼ = 1 Fr. e. |
| 54 = | 24 = | 6 = | 2 = | 1½ = 1½ = 1. |

NOTE.—The Scotch ell contains 4 quarters 1½ inch.

DRY MEASURE.

66. By this are measured all dry wares, as grain, beans, coal, oysters, &c.

The denominations of Dry Measure are chaldrons, bushels, pecks, gallons, quarts, and pints.

TABLE.

| | | | | |
|-------------|--------|-------------|--------|---------|
| 2 pints (1) | make | 1 quart, | marked | qt. |
| 4 quarts | " | 1 gallon, | " | gal. |
| 2 gallons | " | 1 peck, | " | pk. |
| 4 pecks | " | 1 bushel, | " | bu. |
| 36 bushels | " | 1 chaldron, | " | ch. |
| pt. | qt. | | | |
| 2 = | 1 | gal. | | |
| 8 = | 4 = | 1 | pk. | |
| 16 = | 8 = | 2 = | 1 | bu. |
| 64 = | 32 = | 8 = | 4 = | 1 ch. |
| 2304 = | 1152 = | 288 = | 144 = | 36 = 1. |

Our standard of Dry Measure is the Winchester bushel. This is an upright cylinder whose internal diameter is 18½ inches and depth 8 inches. It contains 2150.4 cubic inches of 77.627 lbs. avoirdupois of pure distilled water at 62° Fahr. and 30 in. barometer. The standard unit of Dry Measure in the United States is also the Winchester bushel, so called because the standard measure was formerly kept at Winchester, England. The standard unit of Dry Measure in Great Britain is the Imperial bushel, which is an upright cylinder whose internal diameter is 18.739 inches and depth 8 inches. It contains 2218.192 cubic inches or 80 lbs. avoirdupois of pure distilled water at 62° Fahr. and 30 in. barometer.

Grain is often bought and sold by weight, allowing for a bushel, 60 lbs. of wheat, 56 lbs. of rye, 56 lbs. of Indian corn, 48 lbs. of barley, 34 lbs. of oats, 60 lbs. of peas, 50 lbs. of beans, 40 lbs. of buckwheat, 60 lbs. of timothy or red clover seed.

LIQUID MEASURE.

67. Liquid Measure is used for measuring all liquids. The denominations of Liquid Measure are tuns, pipes, hogsheads, barrels, gallons, quarts, pints, and gills.

TABLE.

| | | | | |
|--------------|--------|-------------|--------|----------------|
| 4 gills (g.) | make | 1 pint, | marked | pt. |
| 2 pints | " | 1 quart, | " | qt. |
| 4 quarts | " | 1 gallon, | " | gal. |
| 31½ gallons | " | 1 barrel, | " | bar. |
| 2 barrels | " | 1 hogshead, | " | hhd. |
| 2 hogsheads | " | 1 pipe, | " | pi. |
| 2 pipes | " | 1 tun, | " | tun. |
| | | | | |
| g. | pt. | qt. | gal. | bar. |
| 4 = | 1 | | | |
| 8 = | 2 = | 1 | | |
| 32 = | 8 = | 4 = | 1 | |
| 1008 = | 252 = | 126 = | 31½ = | 1 hhd. |
| 2016 = | 504 = | 252 = | 63 = | 2 = 1 pi. |
| 4032 = | 1008 = | 504 = | 126 = | 4 = 2 = 1 tun. |
| 8064 = | 2016 = | 1008 = | 252 = | 8 = 4 = 2 = 1. |

The English Imperial gallon contains 277 274 cubic inches or 10 lbs. avoirdupois of pure distilled water, weighed at a temperature of 62° Fahr. and under a barometric pressure of 30 inches.

In the United States the wine gallon contains 231 cubic inches, and the beer gallon 282 cubic inches. The gallon of Great Britain is therefore equal to 1·2 gallons United States Wine Measure.

By an Act of the Imperial Parliament, 1826, the Imperial gallon of 277·274 cubic inches, was adopted as the only gallon, and is therefore the standard for both liquid and dry measure.

Beer is sold usually by the gallon; sometimes, however, in casks of 5 gals., 10 gals., 20 gals., &c. The beer barrel contains 36 gallons, and the hogshead 54 gallons.

TIME MEASURE.

68. Time is naturally divided into days and years—the former measured by the revolution of the earth on its axis, and the latter by the revolution of the earth round the sun.

The denominations of Time Measure are years, months, weeks, days, hours, minutes, and seconds.

TABLE.

| | | |
|--|---------------------------------|-------------|
| 60 seconds (sec.) | make 1 minute, | marked min. |
| 60 minutes | " 1 hour, | " h. |
| 24 hours | " 1 day, | " d. |
| 7 days | " 1 week, | " wk. |
| 4 weeks | " 1 lunar month, | " mo. |
| 13 lunar months or 12 calendar months or 365 $\frac{1}{4}$ days (nearly) | } make 1 civil year, marked yr. | |

| | | | | | |
|------------|----------|--------|---------------------|--------------------|-----|
| sec. | min. | h. | da. | wk. | yr. |
| 60 = | 1 | | | | |
| 3600 = | 60 = | 1 | da. | | |
| 86400 = | 1440 = | 24 = | 1 | wk. | |
| 604800 = | 10080 = | 168 = | 7 = | 1. | yr. |
| 31557600 = | 525960 = | 8766 = | 365 $\frac{1}{4}$ = | 52 $\frac{1}{8}$ = | 1. |

The twelve calendar months, into which the civil or legal year is divided, and the number of days in each, are as follows:

| | | |
|--------------|------------|--------------|
| First month, | January, | has 31 days. |
| Second " | February, | " 28 " |
| Third " | March, | " 31 " |
| Fourth " | April, | " 30 " |
| Fifth " | May, | " 31 " |
| Sixth " | June, | " 30 " |
| Seventh " | July, | " 31 " |
| Eighth " | August, | " 31 " |
| Ninth " | September, | " 30 " |
| Tenth " | October, | " 31 " |
| Eleventh " | November, | " 30 " |
| Twelfth " | December, | " 31 " |

The number of days in the respective months may be recalled by recollecting the following well-known lines:

Thirty days hath September,
April, June, and November;
February has twenty-eight alone,
And all the rest have thirty-one;
But leap-year coming once in four,
February then has one day more.

The number of days in each month may also be recollected by counting the months on the *four* fingers and *three* intervening spaces. Thus, January on the first finger; February in space between first and second fingers; March on second finger; April in second space; May on third finger; June in third space; July on fourth finger; August on first finger (since there are no more spaces); September in first space, &c. Now, when counted thus, all the months having 31 days come on the fingers, and all having 30 only, fall into the spaces.

The solar year is the time elapsing from the passage of the sun from either solstice back to the same again, and is equal to 365d. 5h. 48m. 48sec.

The sidereal year is the time between two successive conjunctions of the sun with some star, and is equal to 365d. 6h. 9m. 14 $\frac{1}{2}$ sec.

The civil or legal year is that in common use among different nations, and is equal to 365 days for three years in succession and to 366 days for the fourth,

This additional day is given to every fourth year in order to make the civil year agree with the solar. It was originally added by repeating the sixth of the calends of March in the Roman calendar—corresponding with the 24th of February with us. The day was called the *intercalary* day, from the Latin *intercalo* to insert; and the year was called *bissextile* from the Latin *bis*, twice, and *sextilis*, sixth, (i. e. sixth calend, taken twice). We now call it Leap Year because it leaps a day more than a common year. This correction was made by Julius Cæsar, emperor of Rome, and hence the civil year is often called the Julian year.

The addition of one day every four years would be strictly correct if the solar year contained 365 days 6h.; but it only contains 365d. 5h. 48m. 48sec., or 11m. 12sec. less than 365d. 6h. Adding one day every four years gives us then an error of excess of 44m. 48sec., or about 3 days for every 400 years. Thus the Julian calendar was behind the solar time—since the Julian year was longer than the natural year. This error at the time of Pope Gregory XIII, amounted to 10 days, which he corrected in 1582 by suppressing 10 days in the month of October, the day after the 4th being called the 15th. Hence this calendar is sometimes called the *Gregorian calendar*.

This correction was not adopted in England till 1752, when the error amounted to 11 days. By *Act of Parliament*, 11 days after the 2d of September were therefore omitted. The *civil* year, by the same act, was made to commence on the 1st of January, instead of the 25th of March, as it had done previously.

Dates reckoned by the *old method* or Julian calendar, are called *Old Style*; and those reckoned by the *new method*, are called *New Style*.

To change any date from *Old* to *New Style*, we must add 11 days to it; and if the given date in *Old Style* is between the 1st of January and the 25th of March, we must add 1 to the year in *New Style*.

Russia still reckons dates according to *Old Style*. The *difference* now amounts to 12 days.

69. To ascertain whether a year is LEAP YEAR.

Divide the given year by 4, and if there is no remainder it is Leap year. The remainder, if any, shows how many years have elapsed since a Leap year occurred.

Thus, dividing the year 1847 by 4, the remainder is 3; hence it is 3 years since the last Leap year, and the ensuing year will be Leap year.

To this rule there is an exception; for we have seen that a *solar* year is 11 min. 12 sec. less than a Julian year, which is 365 $\frac{1}{4}$ days. This error, in 400 years, amounts to about 3 days; consequently, if a day is added every fourth year, that is, if we have 100 leap years in 400 years, according to the Julian calendar, the reckoning would fall three days behind the *solar* time. Thus, reckoning from the commencement of the Christian era, when it was January 1st, 401, by the Julian time, it was January 4th by the *solar* time.

To remedy this error, only 1 *centennial* year in four is regarded a *leap year*; or, which is the same in effect, whenever the *centennial* year, or the *number* expressing the *century*, is not divisible by 4, that year is not a leap year, while the other centennial years are. Thus, 17, 18, 19, denoting 1700, 1800, and 1900, are *not divisible* by 4, consequently they are *not* leap years, though according to the rule above they would be; on the other hand, 16 and 20, denoting 1600 and 2000, are *divisible* by 4, and are therefore leap years. There is still a slight error, but it is so small that in 5000 years it scarcely amounts to a day.

70. TABLE SHOWING THE NUMBER OF DAYS FROM ANY DAY OF ONE MONTH TO THE SAME DAY OF ANY OTHER MONTH IN THE SAME YEAR.

| From any day of | To the same day. | | | | | | | | | | | |
|-----------------|------------------|------|------|--------|-----|------|------|------|-------|------|------|------|
| | Jan. | Feb. | Mar. | April. | May | June | July | Aug. | Sept. | Oct. | Nov. | Dec. |
| January | 365 | 31 | 59 | 90 | 120 | 151 | 181 | 212 | 243 | 273 | 304 | 334 |
| February | 334 | 365 | 28 | 59 | 89 | 120 | 150 | 181 | 212 | 242 | 273 | 303 |
| March | 306 | 337 | 365 | 31 | 61 | 92 | 122 | 153 | 184 | 214 | 245 | 275 |
| April | 275 | 306 | 334 | 365 | 30 | 61 | 91 | 122 | 153 | 183 | 214 | 244 |
| May | 245 | 276 | 304 | 335 | 365 | 31 | 61 | 92 | 123 | 153 | 184 | 214 |
| June | 214 | 245 | 273 | 304 | 334 | 365 | 30 | 61 | 92 | 122 | 153 | 183 |
| July | 184 | 215 | 243 | 274 | 304 | 335 | 365 | 31 | 62 | 92 | 123 | 153 |
| August | 153 | 184 | 212 | 243 | 273 | 304 | 334 | 365 | 31 | 62 | 92 | 122 |
| Sept. | 122 | 153 | 181 | 212 | 242 | 273 | 303 | 334 | 365 | 30 | 61 | 91 |
| October | 92 | 123 | 151 | 182 | 212 | 243 | 273 | 304 | 335 | 365 | 31 | 61 |
| Nov. | 61 | 92 | 120 | 151 | 181 | 212 | 242 | 273 | 304 | 334 | 365 | 30 |
| Dec. | 31 | 62 | 90 | 121 | 151 | 182 | 212 | 243 | 274 | 304 | 335 | 365 |

The months counted from any day of, are arranged in the left-hand vertical column; those counted to the same day of, are in the upper horizontal line: the days between these periods are found in the angle of intersection, in the same way as in a common table of multiplication. If the end of February be included between the two points of time, a day must be added in leap years.

EXAMPLE 1.—How many days are there from the fifteenth of March to the fourth of October? Looking down the vertical row of numbers at the head of which October is placed, and at the same time along the horizontal row at the left hand side of which is March, we perceive in their intersection the number 214:—so many days, therefore, intervene between the fifteenth of March and the fourth of October. But the fourth of October is eleven days earlier than the fifteenth; we therefore subtract 11 from 214, and obtain 203, the number required.

EXAMPLE 2.—How many days are there between the third of January and the nineteenth of May? Looking as before in the table, we find that 120 days intervene between the third of January and the third of May; but as the nineteenth is sixteen days later than the third, we add 16 to 120 and obtain 136, the number required.

Since February is in this case included, if it were a leap year, as that month would then contain 29 days, we should add one to the 136, and 137 would be the answer.

EXAMPLES.

- How many days from May 3d to the 4th of next July?
Ans. 62 days.
- How many days from July 4th to the 25th of next December?
Ans. 174 days.
- How many days from March 21st to the 23rd of the next September?
Ans. 186 days.

4. How many days from September 23rd to the 21st of the next March? *Ans.* 179 days.
5. How many days from June 21st to the 22nd of the next December? *Ans.* 184 days.
6. How many days from December 22nd to the 21st of the next June? *Ans.* 181 days.
7. How many days from March 21st to the 21st of the next June? *Ans.* 92 days.
8. How many days from January 13th, 1848, to September 17th of the same year? *Ans.* 248 days.

71. The *unit of time* is the basis of that of *Length, Mass, and Pressure*: the connections being as follows:—

A Pound Pressure means that amount of pressure which is exerted towards the earth, at the level of the sea, by the *quantity of matter* called a *pound*.

A Pound of Matter means a quantity equal to that quantity of pure water which, at the temperature of 62° Fahr., would occupy 27·272 *cubic inches*.

A cubic inch is that cube whose side, taken 39·1393 times, would measure the effective length of a *London seconds-pendulum*.

A London seconds-pendulum is that which, by the unassisted and unopposed effect of its own gravity, would make 86400 vibrations in an artificial solar day, or 86163·09 in a natural sidereal day.

CIRCULAR MEASURE.

72. Circular Measure, sometimes called Angular Measure, is chiefly used by astronomers, navigators, and surveyors, for measuring angles and for reckoning *latitude* and *longitude*, and the motion of the heavenly bodies.

The Denominations of Circular Measure are signs, degrees, minutes, and seconds.

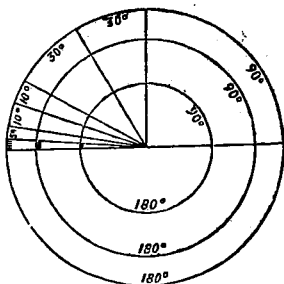
TABLE.

| | | |
|----------------------|----------------|---------------|
| 60 seconds (") | make 1 minute, | marked '. |
| 60 minutes | " 1 degree, | " °. |
| 30 degrees | " 1 sign, | " s. |
| 12 signs or 360 deg. | 1 circle, | " c. |
| " | ' | |
| 60 = | 1 | ° |
| 3600 = | 60 = | 1 s. |
| 108000 = | 1800 = | 30 = 1 c. |
| 1296000 = | 21600 = | 360 = 12 = 1. |

The circumference of every circle is supposed to be divided into 360 equal parts called degrees, as in the subjoined figure. Since a degree is simply the $\frac{1}{360}$ part of the circumference of a circle, it is obvious that its length must depend upon the size of the circle. If the circumference be 360 miles in length, then a degree of that circle will be one *mile* long; if the circle be 360 inches in circumference, then a degree will be one *inch*, &c.

The divisions of the circumference of the circle into 360 equal parts took its origin from the length of the year, which, in round numbers, was supposed to contain 360 days, or 12 months of 30 days each. The 12 *signs* correspond to the 12 months.

The term *minute* is from the Latin *minutum* "a small part." The term *seconds* is an abbreviated expression for *second* minutes, or minutes of the *second order*.



MISCELLANEOUS TABLE.

| | | | |
|-----|----------------------|-------|-----------------------------|
| 73. | 12 individual things | make | 1 dozen. |
| | 12 dozen | | " 1 gross. |
| | 12 gross | | " 1 great gross. |
| | 20 individual things | " | 1 score. |
| | 24 sheets of paper | .. | " 1 quire. |
| | 20 quires | | " 1 ream. |
| | 112 pounds | | " 1 quintal. |
| | 200 " | | " 1 barrel of pork or beef. |
| | 196 " | | " 1 barrel of flour. |

BOOKS.

- A sheet folded into two leaves is called a *folio*.
 " folded into four leaves is called a *quarto*, or 4to.
 " folded into eight leaves is called an *octavo*, or 8vo.
 " folded into twelve leaves is called a *duodecimo*,
 or 12mo.
 " folded into eighteen leaves is called an 18mo.

74. When the figures are written by the side of each other, thus,

2587931272,

the language implies that the *unit* in each place is equivalent to ten units of the place next to the right; or that ten units of any particular place are equivalent to one unit of the place immediately to the left.

75. When figures are written thus,

| | | | |
|----|----|----|----|
| \$ | d. | c. | m. |
| 1 | 4 | 6 | 5 |

the language implies that 10 units of the lowest denomination make one of the second; ten of the second, one of the third; and ten of the third, one of the fourth.

76. When figures are written thus,

| | | | | | |
|----|------|-----|-----|-----|-----|
| T. | cwt. | qr. | lb. | oz. | dr. |
| 16 | 11 | 3 | 21 | 14 | 3 |

the language implies that 16 units of the lowest denomination make one of the second; 16 units of the second, one of the third; 25 units of the third, one of the fourth; 4 of the fourth, one of the fifth; and 20 of the fifth, one of the sixth.

All other denominate numbers are formed on the same principle; and in all of them we pass from a lower to the next higher denomination by considering how many units of the one make one unit of the other.

REDUCTION.

77. Reduction is the changing the denomination of a number from one unit to another, without altering the value of the number. For example, if we desire to reduce 7 of the order of *hundreds* to a lower denomination, we multiply the 7 by 10, and thus obtain 70 of the order *tens*, which are equal to 7 of the *third order* or *hundreds*. If we wish to reduce to a still lower denomination, we multiply the *tens* by ten, and this gives us 700 of the *first order* or *simple units*, which are just equal to 70 *tens* or 7 *hundreds*.

If, on the contrary, we wish to reduce 900 of the *first order* or *simple units* to units of the *third order* or *hundreds*, we divide by 10, and thus obtain 90 of the *second order*, which we again divide by 10 and obtain 9 units of the *third order* or *hundreds*.

Hence reduction of denominate numbers is divided into two parts:—

1st. To reduce a number from a higher denomination to a lower: this is called Reduction Descending.

2nd. To reduce a number from a lower denomination to a higher : this is called Reduction Ascending.

REDUCTION DESCENDING.

EXAMPLE.

78. Reduce £6 16s. 0½d. to farthings.

$$\begin{array}{r}
 \text{£} \quad \text{s.} \quad \text{d.} \\
 6 \quad 16 \quad 0\frac{1}{2} \\
 20 \\
 \hline
 136 \text{ shillings} = \text{£}6 \text{ 16s.} \\
 12 \\
 \hline
 1632 \text{ pence} = \text{£}6 \text{ 16s 0d.} \\
 4 \\
 \hline
 6529 \text{ farthings} = \text{£}6 \text{ 16s 0}\frac{1}{2}\text{d.}
 \end{array}$$

EXPLANATION.—In this example we multiply the £6 by 20, *because* each pound is equal to 20 shillings; 6 pounds are therefore equal to 120 shillings, and the 16 shillings given in the question make 136 shillings. Then we multiply the number of shillings by 12, *because* each shilling is equal to 12 pence, and, since there are no pence in the question, we simply set down the result, 1632 pence. Lastly, we multiply the 1632 pence by 4, *because* each penny is equal to 4 farthings, and to the result we add the one farthing given in the question.

From the above example and solution we deduce the following—

RULE.

Multiply the highest given denomination by that quantity which expresses the number of the next lower contained in one of its units; and add to the product that number of the next lower denomination which is found in the quantity to be reduced.

Proceed in the same way with the result; and continue the process until the required denomination is obtained.

EXERCISE.

1. How many farthings in 23328 pence? *Ans.* 93312.
2. How many shillings in £348? *Ans.* 6960.
3. How many pence in £38 10s.? *Ans.* 9240.
4. How many pence in £58 13s.? *Ans.* 14976.
5. How many farthings in £58 13s.? *Ans.* 56304.
6. How many farthings in £59 13s. 6½d.? *Ans.* 57291.
7. How many pence in £63 0s. 9d.? *Ans.* 15129.
8. How many pounds in 16 cwt., 2 qrs., 16 lb.? *Ans.* 1666.
9. How many pounds in 14 cwts., 3qrs., 16lb.? *Ans.* 1491.
10. How many grains in 3 lb., 5 oz., 12 dwt., 16 grains? *Ans.* 19984.

11. How many grains in 7 lb., 11 oz., 15 dwt., 14 grains? *Ans.* 45974.
12. How many hours in 20 (common) years? *Ans.* 175200.
13. How many feet in 1 mile? *Ans.* 5280.
14. How many minutes in 46 years, 21 days, 8 hours, 56 minutes (not taking leap-years into account)? *Ans.* 24208376.
15. How many square yards in 74 square perches? *Ans.* 2238.5 (2238 and a half.)
16. How many square yards in 46 acres, 3 roods, 12 perches? *Ans.* 226633.
17. How many square acres in 767 square miles? *Ans.* 490880.
18. How many cubic inches in 767 cubic feet. *Ans.* 1325376.
19. How many quarts in 767 pecks? *Ans.* 6136.
20. How many pints in 797 pecks? *Ans.* 12752.

REDUCTION ASCENDING.

79. EXAMPLE.—Reduce 856347 farthings to pounds, &c.

$$\begin{array}{r} 4)856347 \\ \hline 12)214086\frac{3}{4}d. \\ \hline 20)17840s\ 6\frac{3}{4}d. \\ \hline \end{array}$$

$$£892\ 0s.\ 6\frac{3}{4}d.=856347\ \text{farthings.}$$

EXPLANATION.—We divide the farthings by 4, *because* every four farthings are equal to one penny, and it is evident that what remains after taking away four farthings as often as possible from the farthings must be farthings. We thus obtain 856347 farthings, equal to 214086 pence and 3 farthings. Then we divide the pence by 12, *because* every 12 pence are equivalent to one shilling, and what remains after taking 12 pence as often as possible from the pence must be pence. We thus ascertain that 214086 pence are equal to 17840 shillings and 6 pence 3 farthings. Lastly we divide 17840 shillings by 20, *because* every 20 shillings are equal to one pound. By this process we have reduced 856347 farthings to £892 0s. 6¾d.

From the above example and solution we deduce the following—

RULE.

Divide the given number by that number which it takes of the given denomination to make one of the next higher. Set down the remainder, if any, and proceed in the same manner with each successive denomination till you come to the one required. The last quotient, with the several remainders annexed, will be the answer required.

EXERCISE.

1. Reduce 32756 farthings to pounds, shillings, and pence. *Ans.* £34 2s. 5d.
2. Reduce 23547 troy grains to pounds, &c. *Ans.* 4 lb., 1 oz., 1 dwt., 3 grs.

3. Reduce 397024 yards to miles, furlongs, &c.
Ans. 225 m. 4 fur. 26 r. 1 yd.
4. How many hours are there in 28635 seconds?
Ans. 7 h. 57 min. 15 sec.
5. How many cwt., qrs., and pounds in 1666 pounds?
Ans. 16 cwt. 2 qrs. 16 lb.
6. How many cwt, &c. in 1491 pounds?
Ans. 14 cwt. 3 qrs. 16 lb.
7. How many pounds troy in 115200 grains? *Ans.* 20.
8. How many pounds in 107520 oz. avoirdupois? *Ans.* 6720.
9. How many cubic feet, &c. in 1674674 cubic inches?
Ans. 969 feet, 242 inches.
10. How many yards in 767 Flemish ells?
Ans. 575 yards, 1 quarter.
11. How many leagues in 183810 feet?
Ans. 11 lea. 1. m. 6 fur. 20 rd.
12. How many cubic yards in 138297 cubic inches?
Ans. 2 c. yds. 26 ft. 57 in.
13. How many cords of wood are there in 67893 cubic feet?
Ans. 530 cords, 53 cub. ft.
14. In 3561829 seconds how many weeks?
Ans. 5 wks., 6 dys., 5 h., 23 min., 49 sec.
15. In 1597 quarts, how many bushels?
Ans. 49 bushels, 3 pks. 1 gal. 1 qt.
16. In 1000 cord-feet of wood, how many cords?
Ans. 125 cords.
17. In 10,000" how many degrees?
Ans. 2° 46' 40"
18. In 70,000 square links, how many square chains?
Ans. 7 square chains.
19. In 11521 grains apothecaries' weight, how many pounds?
Ans. 2 lbs. 0 $\frac{3}{4}$ 0 $\frac{3}{4}$ 0 $\frac{1}{2}$ 1 gr.
20. In 26025 square feet, how many roods?
Ans. 2 r. 15 sq.p. 17 sq.yds. 8 sq.ft 36 sq. in.

REDUCTION OF THE OLD CANADIAN CURRENCY TO
THE NEW OR DECIMAL CURRENCY.

80. EXAMPLE.—Reduce £76 14s. 10 $\frac{3}{4}$ d. to cents.

$$£76 \times 400 = 30400 \text{ cents.}$$

$$14s \times 20 = 280 \text{ "}$$

$$10\frac{3}{4}d. = 43 \text{ far.} \times 5 \div 12 = 17\frac{11}{12} \text{ "}$$

$$£76 \text{ 14s. } 10\frac{3}{4}d. = 30697\frac{1}{2} \text{ cts.}$$

is equal to 20 cents; and lastly we multiply the number of farthings in the pence and farthings by 5 and divide the result by 12, because each farthing is equal to $\frac{1}{4}$ of a cent.

That each farthing is equal to $\frac{1}{4}$ of a cent is evident from the fact that

EXPLANATION—We multiply £76 by 400, because each pound is equal to 4 dollars or 400 cents; next we multiply 14, the number of shillings, by 20, because each shilling

48 farthings (or one shilling) are equal to 20 cents; or 12 farthings equal 5 cents, or one farthing equal $\frac{1}{12}$ of a cent.

From the above example and solution we deduce the following—

RULE.

Multiply the pounds by 400, the shillings by 20, and take five-twelfths of the number expressing how many farthings there are in the given pence and farthings. Add the three results together and their sum will be the number of cents required.

Consider the last two figures as cents, and the result will be dollars and cents.

NOTE.—We take five-twelfths of the farthings by multiplying them by five and dividing the result by twelve.

EXERCISE.

1. How many cts. are there in £3 7s. 1½d.? *Ans.* 1342½ cts.
2. How many dollars are there in £29 18s. 3½d.?
Ans. 11965½ cents, or \$119.65½ cents.
3. How many cents are there in 11½d.? *Ans.* 18½ cents.
4. How many dollars and cents are there in £69 15s. 6d.?
Ans. 27910 cents, or \$279.10.
5. How many dollars and cents in 18s. 8½d.? *Ans.* \$3.74½.
6. How many dollars and cents in £17 16s. 5½d.?
Ans. \$71.29½.
7. How many dollars and cents in £87? *Ans.* \$348.00.
8. How many dollars and cents in 15s. 11½d.? *Ans.* \$3.19½.
9. How many dollars and cents in £16 6s. 2d.? *Ans.* \$65.23½.
10. Reduce £2 9s. 11d. to dollars and cents. *Ans.* \$9.98½.

RECAPITULATION.

I. Science is a collection of the general *principles* or *leading truths* of any branch of knowledge systematically arranged.

II. Art is a collection of *rules* serving to facilitate the performance of certain operations.

III. The rules of art are based upon the principles of science.

IV. Arithmetic is both a science and an art.

V. The science of arithmetic discusses the *properties* of numbers and the *principles* upon which the elementary operations of arithmetic are founded.

VI. The *science of arithmetic* is called Theoretical Arithmetic.

VII. The *art of arithmetic* is called Practical Arithmetic.

VIII. *Practical Arithmetic* is the *application of rules* based upon the science of numbers, to practical purposes, as the solution of problems, &c.

IX. *Numbers* are expressions for one or more things of the same kind.

X. *Unity*, or the *unit* of a number, is one of the equal things which the number expresses.

XI. Numbers are divided into two classes, viz. : simple or abstract numbers; and applicate, concrete, or denominate numbers.

XII. An applicate, concrete, or denominate number is a number whose unit indicates some particular object or thing.

XIII. A simple or abstract number is a number whose unit indicates no particular object or thing.

XIV. Numbers may be expressed either by words or by characters.

XV. The expression of numbers by characters is called *Notation*.

XVI. The reading of numbers, expressed by characters, is called *Numeration*.

XVII. The characters *we* use to express numbers are either *letters* or *figures*.

XVIII. The expression of numbers by *letters* is called *Roman Notation*.

XIX. The expression of numbers by *figures* is called *Arabic Notation*.

XX. In the *Roman Notation* only seven numeral letters are used, viz. : I, V, X, L, C, D, M.

XXI. When these letters stand alone, I denotes *one*, V *five*, X *ten*, L *fifty*, C *one hundred*, D *five hundred*, M *one thousand*.

XXII. All other numbers are expressed by repetitions and combinations of these letters.

XXIII. In combinations of these numerical letters, every time a letter is repeated its value is repeated; also when a letter of a lower value stands *before* one of a higher, its value is to be *subtracted*; but when a letter of a lower comes directly *after* one of a higher value, its value is to be added.

XXIV. A bar or dash written over a letter or combination of letters, multiplies its value by one thousand. As we have already a character for one thousand, viz., M, and can, by repeating it, express *two* or *three thousand*, we do not dash the I, or combinations into which it enters.

XXV. Anciently, IV was written IIII; IX was written VIIII; XL was written XXXX, &c.; D was written I \bar{D} , and M was written CI \bar{D} . Affixing C to I, increases its value ten times—thus I \bar{D} =500; I \bar{D} D=5000; I \bar{D} DD=50000, &c. Prefixing C and affixing D to CI \bar{D} increases its value also ten times, thus, CI \bar{D} =1000; CCI \bar{D} D=10000; CCCI \bar{D} DD=100,000, &c.

XXVI. The figures or characters used in the Arabic or common system of notation are 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, one, two, three, four, five, six, seven, eight, nine, zero.

XXVII. The first nine of these characters are called *significant figures*, because each one has always some value, or denotes some number. They are also called *digits* (Lat. *digitus*, "a finger"), from the almost universal habit of counting on the *fingers*.

XXVIII. The last or *zero* is called a *cipher* or *naught*, because it is *valueless*, that is, stands for *nothing*. It is not, however, *useless*, since it serves to give the significant figures their appropriate places.

XXIX. When the 0 stands to the left of an integral number, or to the right of a decimal, i. e. when it does not come *between* the decimal point and some significant figure, it is both *valueless* and *useless*.

XXX. The digits 1, 2, 3, &c. standing immediately to the left of the *decimal point*, expressed or understood, are called *simple units*, or *units of the first order*.

XXXI. The *decimal point* is a small dot or point, used to indicate the *position* of the *simple units*.

XXXII. The digits 1, 2, 3, &c. standing one place to the left of the simple units, are called *tens*, or units of the *second order to the left*. When they stand one place to the right of the simple unit, they are called *tenths*, or units of the *second order to the right*.

XXXIII. The digits 1, 2, 3, &c. when standing two places to the left of the simple unit, are called *hundreds*, or units of the *third order to the left*. When standing two places to the right, they are called *hundredths*, or units of the *third order to the right*, &c.

XXXIV. Commencing at the simple units and proceeding to the *left*, we have *units* of the *first order* or *simple units*; next, units of the *second order* or *tens*; next, units of the *third order* or *hundreds*; next, units of the *fourth order* or *thousands*; next, units of the *fifth order* or *tens of thousands*, &c.

XXXV. Commencing at the simple units and proceeding to the *right*, we have units of the *first order* or *simple units*; next, units of the *second order* or *tenths*; next, units of the *third order* or *hundredths*; next, units of the *fourth order* or *thousandths*; next, units of the *fifth order* or *tenths of thousandths*, &c.

XXXVI. Each digit has two values, viz.: a simple or absolute value, and a local or relative value.

XXXVII. The *simple* or *absolute value* of a digit is the value it expresses when simply considered as representing a certain number of repetitions of the digit *one*.

XXXVIII. The *local* or *relative value* of a digit is the value it expresses when considered as occupying a certain *position* with reference to the decimal point.

XXXIX. The *ratio* of one number to another is the *relation* which one bears to the other with respect to magnitude, when the comparison is made by considering, not by *how much* the one is greater or less than the other, but *what number of times* it contains it, or is contained in it.

XL. When several numbers, or groups of units, are so arranged that the second and third have the same ratio to one another as the first and second, and the third and fourth the same ratio as the second and third, &c.,—they (the numbers or groups of units) are said to have a *common ratio*.

XLI. The common ratio of our system of numbers is 10—by saying which we merely mean that the different orders increase or decrease from one another in a ten-fold

proportion, i. e. that 10 units of any one order make one unit of the next higher, and *vice versâ*.

XLII. A system of numbers is called a *binary, ternary, quaternary, quinary, senary, septenary, octenary, nonary, denary, &c.* system, according as *two, three, four, five, six, seven, eight, nine, or ten* is the common ratio of the orders. Ours is a denary or decimal system.

XLIII. To facilitate the reading of a number we divide it into *periods* of three places each, by placing separating points after every third figure right and left of the decimal point.

XLIV. The periods to the left of the decimal point are *units, thousands, millions, billions, trillions, &c.* The periods to the right of the decimal point are *thousandths, millionths, billionths, trillionths, &c.*

XLV. The *lowest order* used in any reading, whether it be thousands, units, hundredths, tenths of thousandths, hundredths of millionths, &c., gives the name or denomination to the part or whole of the number used in the reading.

XLVI. Numbers to the left of the decimal point are called *integers* or whole numbers; those to the right of the decimal point are called *decimals*.

XLVII. A number is multiplied by 10 every time the decimal point is moved *one* place to the right, and divided by 10 every time the decimal point is moved *one* place to the left. Thus, moving the decimal point *two, four, or six* places, either multiplies or divides the number by 100, 10,000, or 1,000,000, according as we move it to the right or to the left.

XLVIII. A number may be read in several ways by changing the nature of the simple unit. Thus the number 576.24 may be read :

- 1st. Five hundreds, seven tens, six units, two tenths, and four hundredths.
- 2nd. Fifty-seven tens, six units, two tenths, and four hundredths.
- 3rd. Five hundred and seventy-six units, two tenths, and four hundredths.
- 4th. Five thousand, seven hundred and sixty-two tenths, and four hundredths.
- 5th. Fifty-seven thousand, six hundred and twenty-four hundredths.
- 6th. Five hundred and seven thousands, six hundred and twenty-four hundredths.

- 7th. Fifty-seven tens, and six hundred and twenty-four hundredths.
 8th. Five hundred and seventy-six units, and twenty-four hundredths.
 9th. Fifty-seven tens, sixty-two tenths, and four hundredths.
 10th. Five hundreds, seven hundred and sixty-two tenths, and four hundredths, &c.

EXERCISE.

MISCELLANEOUS PROBLEMS.

1. Reduce 6789634 links to acres, and prove by reducing the result to links.
2. Read 67845398678904 and 5900704060040000.00060604.
3. Set down 4769 in Roman numerals.
4. Make 42986 ten thousand times greater.
5. Reduce £16 16s. 6½d. Old Canadian Currency to Dollars and Cents.
6. Read LXXVMMCMXCI.
7. Write down, in Arabic numerals, six hundred and five billions, seventy thousand and sixteen, and nine millionths.
8. Make 469789 one hundred times greater.
9. Read the number 6798 in *all* the ways it can be read. (See Recapitulation XLVIII.)
10. Divide 69800463 by *one million*.
11. Divide 8439 by ten thousand.
12. Multiply 6789 by one hundred thousand.
13. Multiply 60432986 by ten millions.
14. Write down one quadrillion one billion one thousand and one, and one trillionth.
15. Write down seven thousand six hundred and nine tenths of millionths.
16. Read 90807060504030 and
4004040000060432.01010203040506
17. Reduce 6789463 inches to acres, and prove by reducing the result to inches.
18. Reduce 617 cord-feet of wood to cords.
19. Reduce 91867 cubic feet of wood to cords.

20. Write down 718, 614, 499, 999, 8643, 96149, 163986, and 444444 in Roman numerals.

21. Read CCCXXXIII, MCMLXXXIX, and \bar{M} .

22. Read 6129 in as many ways as it can be read.

23. Give all the readings of 634986.

24. Give all the readings of 19'639.

25. Reduce 18s. 9½d., £6 2s. 11d.; 3s. 7d.; and £189 7s. 4½d. to dollars and cents.

26. Give all the readings of the number \$69.863 Federal money.

27. Give all the readings of 9 bush. 3 pk. 1 gal. 3 qts. 1 pt.

28. Were the years 1693, 1856, 1728, 1549, 867, 444, 1600, and 927, leap years or not? If not, how many years after or before leap year?

29. How many days from this to the 17th of next March?

30. Answer the following questions; What is the meaning of the symbols £ s. d. and q.? In the expression " $\frac{\%}{\%}$ " what does the long mark (/) represent? What is the derivation of the word sterling? Why are the pound and guinea so called? What is the derivation of the sign \$? What is the derivation of the words "grain," "pennyweight," "ounce," and "inch"? What is a "carat"? What is a square? Show that a square yard contains 9 square feet. Show that a cubic yard contains 27 cubic feet. What is a cubic yard? What is meant by a ton of round timber? What must be the dimensions of a pile of wood in order that it shall contain a cord? What is meant by a *cord-foot*? What are the dimensions of the *Imperial bushel*?—of the *Winchester bushel*? Which of these is our standard? Which that of the United States? How many pounds of wheat go to the bushel?—of rye?—of oats?—of barley?—of peas?—of beans?—of buckwheat?—of Indian corn? What is our standard for liquid measure? How many cubic inches of water are there in the Imperial gallon? How many pounds Avoirdupois? What are the standard gallons of the United States? Explain why a day is added to every fourth year. What is the origin of the divisions of the circle into degrees and signs? What is the derivation of the terms "minute" and "second"? How many sheets of paper are there in a quire? How many quires in a ream? How many pounds are there in a barrel of flour? What is the meaning of folio?—of 4to or quarto?—of 8vo or octavo?—of 12mo or duodecimo?—of 16mo?—of 18mo?

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—Numbers in Roman numerals, thus, XVI, refer to the articles in the recapitulation; those in Arabic numerals, thus 16, refer to the numbered articles of the Section.

- | | |
|--|--|
| 1. What is science? (I.) | 2. What is art? (II.) |
| 3. Upon what are the rules of art based? (III.) | 4. Is arithmetic a science or an art? (IV.) |
| 5. What are the objects of the science of arithmetic? (V.) | 6. What is the science of arithmetic called? (VI.) |
| 7. What name is given to the art of arithmetic? (VII.) | 8. What is practical arithmetic? (VIII.) |
| 9. What are numbers? (IX.) | 10. What is the unit of a number? (X.) |
| 11. How many classes of numbers are there? (XI.) | 12. What are applicate or denominative numbers? (XII.) |
| 13. What are simple or abstract numbers? (XIII.) | 14. By how many methods may numbers be expressed? (XIV.) |
| 15. What is Notation? (XV.) | |
| 16. What is Numeration? (XVI.) | |
| 17. What characters do we use to express numbers? (XVII.) | |
| 18. What is Roman Notation? (XVIII.) | |
| 19. What is Arabic Notation? (XIX.) | |
| 20. What numeral letters are used in Roman Notation? (XX.) | |
| 21. What is the value of each of these letters when standing alone? (XXI.) | |
| 22. How are all other numbers expressed in Roman Notation? (XXII.) | |
| 23. In combination, when a letter is repeated, what does it indicate? (XXIII.) | |
| 24. When a letter of a lower is placed before one of a higher value, what does it indicate? (XXIII.) | |
| 25. When a letter of a lower is placed after one of a higher value, what does it indicate? (XXIII.) | |
| 26. What effect has a bar or dash written over a letter or expression? (XXIV.) | |
| 27. How do we always write 1000, 2000, 3000? (XXIV.) | |
| 28. Why do we not dash the I or expressions into which it enters? (XXIV.) | |
| 29. How were four, nine, forty, &c., anciently written? (XXV.) | |
| 30. How were 500 and 1000 anciently written? (XXV.) | |
| 31. How were the expressions IQ and CIQ increased in value in ten-fold proportion? (XXV.) | |
| 32. What are the characters used in Arabic or Common Notation? (XXVI.) | |
| 33. What are significant figures, and why are they so called? (XXVII.) | |
| 34. What are digits, and why are they so called? (XXVII.) | |
| 35. Why is 0 called "cipher" or "naught"? (XXVIII.) | |
| 36. Is the cipher of any value? Is it of any use? (XXVIII.) | |
| 37. When is the cipher or 0 both valueless and useless? (XXIX.) | |
| 38. When are digits called simple units or units of the first order? (XXX.) | |
| 39. What is the decimal point? (XXXI.) | |
| 40. When are digits called tens or units of the second order to the left? (XXXII.) | |
| 41. When are digits called tenths, or units of the second order to the right? (XXXII.) | |
| 42. When are digits called hundreds, thousands, hundredths, thousandths, &c.? (XXXIII.) | |
| 43. Name the different orders to the left of the decimal point?—to the right? (XXXIV.) (XXXV.) | |
| 44. How many values has each digit? What are they? (XXXVI.) | |
| 45. What is the simple or absolute value of a digit? (XXXVII.) | |
| 46. What is the local or relative value of a digit? (XXXVIII.) | |
| 47. What is meant by the ratio one number bears to another? (XXXIX.) | |
| 48. What is meant by a common ratio? (XL.) | |

49. What is meant by saying that 10 is the common ratio of our *system of numbers*? (XLI.)
50. What name is given to a system having 10 for its common ratio?—to one having six?—to one having 8?—to one having 2?—to one having 12?—to one having 7? (XLII.)
51. Why are periods used? How many places are there in each period? (XLIII.)
52. Name the periods right and left of the decimal point? (XLIV.)
53. What order gives the name or denomination to the number read? (XLV.)
54. What are *integers*? What are *decimals*? (XLVI.)
55. How does it affect a number to remove the decimal point to the right? How to remove it to the left? (XLVII.)
56. How may a number be read in several ways? (XLVIII.)
57. When figures are written thus, 673'32, what does the notation imply? (Article 74.)
58. When figures are written thus, 6 d. 23 h. 16 min. 37 sec., what does the notation imply? (75 and 76.)
59. What is Reduction? (77.)
60. Into what two parts is Reduction divided? (77.)
61. What is Reduction Descending? Give an example. (77.)
62. What is Reduction Ascending? Give an example. (77.)
63. Give the rule for Reduction Descending. (78.)
64. Give the rule for Reduction Ascending. (79.)
65. What are the denominations of Sterling money? Give the table. (54.)
66. How are pounds, shillings, and pence reduced to farthings? Give the process and the reason for each step. (Answer this and similar succeeding questions after the following *model*.) We multiply the pounds by twenty, and add in the shillings, *because* each pound is equal to twenty shillings. We multiply the shillings by twelve and add in the pence, *because* each shilling is equal to twelve pence. And lastly, we multiply the pence by four and add in the farthings, *because* each penny is equal to four farthings.
67. What are the denominations of Federal money? Give the table. (55.)
68. What are the denominations of Canadian money, old currency? Give the table. (56.)
69. What are the denominations of Canadian money, new currency? Give the table. (57.)
70. How is Old Canadian Currency reduced to New? Give the process and reasons for each step. (58.)
71. What are the denominations of Avoirdupois weight? Give the table. (58.)
72. How many pounds are there in the new cwt.? How many in the old cwt.? (58.)
73. How are tons reduced to drams? (58 and 78.)
74. What are the denominations of Troy weight? Give the table. (59.)
75. How are grains Troy reduced to pounds Troy? Give the process and reason for each step. (59 and 79.) (Answer this and succeeding similar questions after the following *model*.) We divide the grains by 24, *because* every 24 grains are equal to one pennyweight. We divide the resulting pennyweights by 20, *because* every 20 pennyweights are equal to one ounce. And lastly, we divide the resulting ounces by 12, *because* every 12 ounces are equal to one pound.
76. What are the denominations of Apothecaries' weight? Give the table. (60.)
77. How are pounds, ounces, &c., Apothecaries' weight, reduced to grains? (60 and 78.) Answer as in question 66.
78. What are the denominations of Long measure? Give the table. (61.)
79. How are lines reduced to leagues? (61 and 79.) Answer after model in question 75.
80. What are the denominations of Square measure? Give the table. (62.)
81. How are square miles reduced to square inches? (62 and 78.) Answer after model.
82. How are links reduced to acres? (63 and 79.) Answer after model.

83. What are the denominations of Solid measure? Give the table. (64.)
 84. How are cubic inches reduced to cubic feet? (64 and 79.)
 85. How are cubic feet of wood reduced to cords? (64 and 79.)
 86. What is a *cord-foot*? (64.)
 87. What are the denominations of cloth measure? Give the table. (65.)
 88. How are English ells reduced to inches? Answer after model.
 89. What are the denominations of Dry measure? Give the table. (66.)
 90. How are pints reduced to chaldrons? Answer after model.
 91. What are the denominations of Liquid measure? Give the table. (67.)
 92. How are tuns reduced to gills? Answer after model.
 93. What are the denominations of Time measure? Give the table. (68.)
 94. How are seconds reduced to years? Answer after model. (68.)
 95. Name the months, and the number of days in each. (68.)
 96. What is the Solar year and its length?—the Sidereal year and its length?—the Civil year and its length? (68.)
 97. How can we ascertain whether any given year be Leap-year? (69.)
 98. Show that the unit of time is the basis of the units of length, mass or capacity, and weight. (71.)
 99. What are the denominations of Circular measure? Give the table. (72.)
 100. Upon what does the length of a degree depend? How are degrees reduced to seconds?

SECTION II.

FUNDAMENTAL RULES.

1. Arithmetic may be divided into four parts :—
 - 1st. The Arithmetic of Whole Numbers, or that which treats of the properties of entire units.
 - 2nd. The Arithmetic of Fractions, or that which treats of the parts of units.
 - 3rd. The Arithmetic of Ratios, which treats of the relations of numbers, whether integral or fractional, to each other and to the unit 1.
 - 4th. The Application of Arithmetic to practical and useful purposes.
2. The Arithmetic of Whole Numbers includes Addition, Subtraction, Multiplication, Division, Involution, Evolution, &c.
3. The Arithmetic of Fractions may be divided into two parts :—
 - 1st. Vulgar or Common Fractions, in which the unit is *divided into any number of equal parts*.
 - 2nd. Decimal Fractions in which the unit is divided according to the *scale of ten*.
4. The Arithmetic of Ratios relates to the comparison of numbers with respect to their quotients, and embraces Proportion and Progression.

5. Addition, Subtraction, Multiplication, Division, are called the *fundamental rules*, or *ground rules* of Arithmetic, because all the other operations of Arithmetic are performed by means of them.

6. Whatever operations we may perform upon a number, we can only either *increase it* or *diminish it*. If we increase it, the process belongs to addition; if we diminish it, to subtraction. All the rules of arithmetic are therefore resolvable into these two. Multiplication is only a short method of performing a peculiar kind of addition, in which the addends are all the same; and division is merely an abridged method of performing a particular kind of subtraction, in which the same quantity is to be taken away from a given number as often as possible.

When *any* number of quantities, either *different*, or *repetitions* of the same, are united together so as to form but one, we term the process, simply, "Addition." When the quantities to be added are the *same*, but we may have *as many* of them as we please, it is called "Multiplication;" when they are not only the *same*, but their number is indicated by *one of them*, the process belongs to "Involution." That is, addition restricts us neither as to the kind, nor the number of the quantities to be added; multiplication restricts us as to the kind, but not the number; involution restricts us both as to the kind and number. All, however, are really comprehended under the same rule—*addition*.

ADDITION.

7. The sum of two or more numbers is a number which contains as many units, and no more, as are found in all the given numbers.

8. Addition is the process of finding the sum of two or more numbers.

9. The quantities to be added together are called *addends*, and the result of the addition is called the *sum* of the addends.

10. Only those quantities can be added which have the same unit, or, in other words, which are of the same denomination.

Thus it is evident that 6 days and 7 miles cannot be added, since the result would neither be 13 days nor 13 miles; nor can 5 shillings and 3 pence be added, as the result would neither be shillings nor pence. Similarly, we cannot add units and tens, or tenths and hundredths, or units and sevenths, &c.

11. Hence, in writing down the addends preparatory to adding, we must be careful to set units of the same denomination in the same vertical column, *i. e.* units under units, tens under tens, hundreds under hundreds, &c.; shillings under shillings, pence under pence, &c.; miles under miles, furlongs under furlongs, rods under rods, &c.

EXERCISE.

| | |
|--|--|
| $\begin{array}{r} \text{(1)} \\ \text{Apples.} \\ \text{Addends } \left\{ \begin{array}{l} 2 \\ 3 \\ 2 \end{array} \right. \\ \hline \text{Sum of Addends } 7 \end{array}$ | $\begin{array}{r} \text{(2)} \\ \text{Shillings.} \\ \text{Addends } \left\{ \begin{array}{l} 9 \\ 8 \\ 7 \end{array} \right. \\ \hline \text{Sum of Addends } 24 \end{array}$ |
| $\begin{array}{r} \text{(3)} \\ \text{Addends } \left\{ \begin{array}{l} 9 \\ 7 \\ 6 \\ 8 \end{array} \right. \\ \hline \text{Sum of the Addends } 30 \end{array}$ | |
| $\begin{array}{cccccccc} \text{(4)} & \text{(5)} & \text{(6)} & \text{(7)} & \text{(8)} & \text{(9)} & \text{(10)} & \text{(11)} \\ \text{cwt.} & \text{pence.} & \text{sevenths.} & \text{horses.} & \text{tens.} & \text{millionths.} & \text{\$} & \text{miles.} \\ \hline 9 & 4 & 6 & 1 & 7 & 6 & 9 & 7 \\ 6 & 7 & 5 & 9 & 8 & 9 & 8 & 1 \\ 9 & 8 & 4 & 8 & 9 & 8 & 1 & 2 \\ 8 & 9 & 3 & 7 & 6 & 3 & 2 & 3 \\ 7 & 6 & 5 & 5 & 5 & 2 & 3 & 4 \\ \hline 39 & 34 & 23 & 30 & 35 & 28 & 23 & 17 \end{array}$ | |

12. Let it be required to add together 987 and 689.

| | | | | |
|------|------|------|------|------|
| I. | II. | III. | IV. | V. |
| 987 | 987 | 987 | 987 | 987 |
| 689 | 689 | 689 | 689 | 689 |
| 1500 | 160 | 16 | 16 | 1676 |
| 160 | 1500 | 160 | 16 | |
| 16 | 16 | 1500 | 15 | |
| 1000 | 70 | 6 | 1676 | |
| 600 | 600 | 70 | | |
| 70 | 6 | 600 | | |
| 6 | 1000 | 1000 | | |
| 1676 | 1676 | 1676 | | |

EXPLANATION.—We place the given numbers, 987 and 689, under each other, according to (11) and draw a line to separate the addends from the sum.

It is manifest that so long as we add the units of the several orders it is quite immaterial whether we commence at the highest, at the lowest, or at an intermediate denomination.

In the first of the above operations we have commenced continually at the highest or left-hand order. The hundreds added make 15 hundreds or one thousand and five hundred, which we set down; the tens added make 16 tens, equal to 1 hundred and 6 tens, and the units added, make 16 units, equal to 1 ten and 6 units, all of which we set down in their appropriate columns.

Next considering the partial sums 1500, 160, and 16, as so many new addends, we proceed similarly with them and obtain a new set of partial sums, viz: 1000, 600, 70 and 6. But, from the principles of notation (Sec. I). these last numbers (*i. e.* 1000, 600, 70 and 6) may be written in one line, thus, 1676, which therefore, is the sum of the addends 987 and 589.

In (II), (III), (IV), (V) the same result is obtained by a slightly different process.

In (II) we have commenced at the *tens*, and in (III), (IV) and (V) at the units or lowest order. (IV) is simply (III) with the unnecessary 0's omitted.

(V) is (IV) somewhat modified as follows:—9 *units* and 7 *units* make 16 *units*, equal to 6 *units*, which we set down, and one *ten* which we carry to the next column or column of tens; 1 *ten* and 8 *tens* make 9 *tens*, and 8 *tens* make 17 *tens*, equal to 7 *tens*, which we set down, and 1 *hundred*, which we carry to the column of hundreds; 1 *hundred* and 6 *hundreds* make 7 *hundreds*, and 9 *hundreds* make 16 *hundreds*, equal to 6 *hundreds* and 1 *thousand*, both of which we set down.

13. From (I), (II), and (III), it is manifest that it is *as legitimate* to commence at the lowest denomination as at the highest: and from (IV) and (V), that it is *most convenient* to commence at the lowest denomination.

14. From (V) we learn that when we have obtained the sum of the units, in any column, we reduce it to the next higher denomination, and, setting down the remainder under the column added, carry the units of the next higher denomination to their proper column.

15. The reasoning in (12), (13) and (14) applies to any numbers whatever, whether abstract or denominate, and from it, for addition, we deduce the following general—

RULE.

Write down the numbers so that units of the same denomination shall fall in the same column (Arts. 10 and 11).

Draw a line beneath the addends (Art. 12).

Add up the units of the lowest denomination and divide their sum by so many as make one of the denomination next higher (Arts. 13 and 14).

Set down the remainder and carry the quotient to the next higher denomination (Art. 14.)

Proceed in the same manner through all the denominations to the last.

16. We commence at the lowest order or tenths of thousandths. There

| | |
|-----------|---|
| EXAMPLE. | being nothing to add to the 9 tenths of thousandths |
| 698.9649 | we simply set down the 9 in its appropriate column. |
| 84.76 | Next we add the thousandths, thus:— 2 thousandths |
| 9.896 | and 6 thousandths are 8 thousandths and 4 thousandths |
| 98.462 | are 12 thousandths, which are equal to 2 thousandths |
| 989.9 | and 1 hundredth. The 2 thousandths we write down |
| ----- | in its own column and carry the hundredth to the |
| 1881.9829 | column of hundredths. Next we add the column of |

hundredths, thus:—1 hundredth (carried) and 6 hundredths make 7 hundredths and 9 hundredths make 16 hundredths, and 6 hundredths make 22 hundredths and 6 hundredths make 28 hundredths which are equal to 8 hundredths and two tenths. We set down the 8 hundredths and carry the two tenths to the next column or column of tenths. Adding the tenths we find their sum to be 39 tenths, equal to 9 tenths, which we set down, and 3 units which we carry. The simple units added make 41 units, equal to 1 unit, which we set down and 4 tens which we carry; the tens added make 38 tens, equal to 8 tens and 3 hundreds; the hundreds added (with the 3 hundreds we carry) make 18 hundreds, or 8 hundreds, and 1 thousand, both of which we set down in their proper columns.

17. We commence as in (16) with the lowest denomination, which, in

| | |
|----------|--|
| EXAMPLE. | this example, is cents. 89 cents and 42 cents, and 56 |
| \$69.89 | cents and 89 cents, added, make 376 cents. But every |
| 11.56 | 100 cents make one dollar, 376 cents are therefore equal |
| 73.42 | to 3 dollars and 76 cents. The 76 cents we set down in |
| 91.89 | their proper place and carry the 3 dollars to the column |
| ----- | of dollars. |

\$246.76

18. EXAMPLE.—Add together £52 17s. 3½d., £47 5s. 6½d., and £66 14s. 2½d.

| | | | | |
|----|----|----|---|----------|
| £ | s. | d. | } | addends. |
| 52 | 17 | 3½ | | |
| 47 | 5 | 6½ | | |
| 66 | 14 | 2½ | | |
| | | | | |

£166 17 0½ sum.

½ and ½ make 3 farthings, which, with ½, make 6 farthings; these are equivalent to one of the next denomination, or that of pence, to be carried, and two of the present, or one half-penny, to be set down. 1 penny (to be carried) and 2 are 3, and 6 are 9, and 3 are 12 pence—equal to one of the next denomination, or that of shillings, to be carried, and no pence to be set down; we therefore put a cypher in the pence place of the sum. 1 shilling (to be carried) and 14 are 15, and 5 are 20, and 17 are 37 shillings—equal to one of the next denomination, or that of pounds, to be carried, and 17 of the present, or that of shillings, to be set down. 1 pound and 6 are 7, and 7 are 14, and 2 are 16 pounds,—equal to 6 units of pounds, to be set down, and 1 ten of pounds to be carried; 1 ten and 6 are 7 and 4 are 11 and 5 are 16 tens of pounds, to be set down.

When the addends are very numerous, we may divide them into two or more parts by horizontal lines, and, adding each part separately, may afterwards find the amount of all the sums.

| £ s. d. | | | EXAMPLE. | | | | | | |
|---------|----|---|----------|----|---|----|----|---|--|
| 57 | 14 | 2 | } | -- | £ | s. | d. | } | |
| 32 | 16 | 4 | | | | | | | |
| 19 | 17 | 6 | | | | | | | |
| 8 | 14 | 2 | | | | | | | |
| 32 | 5 | 9 | | | | | | | |
| 47 | 6 | 4 | } | = | £ | s. | d. | } | |
| 32 | 17 | 2 | | | | | | | |
| 56 | 3 | 9 | | | | | | | |
| 27 | 4 | 2 | | | | | | | |
| 52 | 4 | 4 | | | | | | | |
| 37 | 8 | 2 | | | | | | | |

Or, in adding each column, we may put down an asterisk, thus *, as often as we come to a quantity which is at least equal to that number of the denomination added which is required to make one of the next—carrying forward what is above this number, if anything, and putting the last remainder, or—when there is nothing left at the end—a cypher under the column;—we carry to the next column one for every asterisk. Using the same example :

| £ | s. | d. |
|-----------|-----|----|
| 57 | *14 | 2 |
| 32 | 16 | 4 |
| 19 | *17 | *6 |
| 8 | *14 | 2 |
| 32 | 5 | *9 |
| 47 | *6 | 4 |
| 32 | 17 | 2 |
| 56 | *3 | *9 |
| 27 | 4 | 2 |
| 52 | 4 | 4 |
| 37 | 8 | 2 |
| 404 11 10 | | |

2 pence and 4 are 6, and 2 are 8, and 9 are 17 pence—equal to 1 shilling and 5 pence; we put down a dot or an asterisk and carry 5. 5 and 2 are 7, and 4 are 11, and 9 are 20 pence—equal to 1 shilling and 8 pence; we put down a dot or an asterisk and carry 8. 8 and 2 are 10 and 6 are 16 pence equal to 1 shilling and 4 pence; we put down a dot and carry 4. 4 and 4 are 8 and 2 are 10—which being less than 1 shilling, we set down under a column of pence to which it belongs, &c. We find on adding them up, that there are three dots; we therefore carry 3 to the column of shillings. 3 shillings and 8 are 11, and 4 are 15, and 4 are 19, and 3 are 22 shillings—equal to 1 pound and 2 shillings; we put down a dot and carry 2. 2 and 17 are 19, &c.

Care is necessary, lest the dots, not being distinctly marked, may be considered as either too few or too many. This method though now but little used, seems a convenient one.

PROOF OF ADDITION.

10. FIRST METHOD.—Go through the process again, beginning at the top and adding downwards.

This method of proof is merely doing the same work twice, in a slightly different manner.

SECOND METHOD.—Separate the addends into two parts. Add each part separately, in the usual way, and then add their sums. If the last sum is the same as that found by the first addition, the work may be presumed to be correct.

This method of proof is founded on the axiom that “the whole is equal to the sum of all its parts.”

EXAMPLE.—Find the sum of 509267, 235809, 72910, and 83925.

| OPERATION. | PROOF BY SECOND METHOD. | |
|------------|---------------------------|--------|
| 509267 | 509267 | 72910 |
| 235809 | 235809 | 83925 |
| 72910 | | |
| 83925 | Partial sums 745076 | 156835 |
| Sum 901911 | First partial sum..745076 | |
| | Second partial sum 156835 | |

Proof..... 901911

EXERCISE.

| (12) | (13) | (14) | (15) | (16) | (17) |
|----------|----------|-------|--------|----------|---------|
| Dollars. | Bushels. | Days. | Acres. | Dollars. | Pounds. |
| 15 | 76 | 765 | 392 | 5832 | 98764 |
| 26 | 48 | 381 | 446 | 8907 | 8753 |
| 18 | 59 | 872 | 872 | 4671 | 76 |
| 61 | 81 | 315 | 969 | 6789 | 9889 |
| 120 | 264 | 2333 | 2679 | 26199 | 117482 |

(18—41)

The sum of the numbers in each row of the following table, whether taken vertically or horizontally, or from corner to corner, is 24156. Let the pupil be required to make these 24 distinct additions.*

TABLE.

| | | | | | | | | | | |
|------|------|------|------|------|------|------|------|------|------|------|
| 2016 | 4212 | 1656 | 3852 | 1296 | 3492 | 936 | 3132 | 576 | 2772 | 216 |
| 252 | 2052 | 4248 | 1692 | 3888 | 1332 | 3528 | 972 | 3168 | 612 | 2412 |
| 2448 | 288 | 2088 | 4284 | 1728 | 3924 | 1368 | 3564 | 1008 | 2808 | 648 |
| 684 | 2484 | 324 | 2124 | 4320 | 1764 | 3960 | 1404 | 3204 | 1044 | 2844 |
| 2880 | 720 | 2520 | 360 | 2160 | 4356 | 1800 | 3600 | 1440 | 3240 | 1080 |
| 1116 | 2916 | 756 | 2556 | 396 | 2196 | 3996 | 1836 | 3636 | 1476 | 3276 |
| 3312 | 1152 | 2952 | 792 | 2592 | 36 | 2232 | 4032 | 1872 | 3672 | 1512 |
| 1548 | 3348 | 1188 | 2988 | 432 | 2628 | 72 | 2268 | 4068 | 1908 | 3708 |
| 3744 | 1584 | 3384 | 828 | 3024 | 468 | 2664 | 108 | 2304 | 4104 | 1944 |
| 1980 | 3780 | 1224 | 3420 | 864 | 3060 | 504 | 2700 | 144 | 2340 | 4140 |
| 4176 | 1620 | 3816 | 1260 | 3456 | 900 | 3096 | 540 | 2736 | 180 | 2376 |

* This table is formed by multiplying the numbers in the magic square of 11 by 36.

| | | | | | |
|--------------|------|-------|-------|-------|-------|
| (42) | (43) | (44) | (45) | (46) | (47) |
| 74564 | 5676 | 76746 | 67674 | 42·37 | 0·87 |
| 7674 | 1567 | 71207 | 75670 | 56·84 | 5·273 |
| 376 | 63 | 100 | 36 | 27·92 | 8·127 |
| 6 | 6767 | 56 | 77 | 62·41 | 25·63 |
| <u>82620</u> | | | | | |

| | | | |
|---------------|---------|---------|--------|
| (48) | (49) | (50) | (51) |
| 3·785 | 85·742 | 0·00007 | 5471·3 |
| 20·766 | 6034·82 | 0·06236 | 563·47 |
| 0·253 | 57·8563 | 0·0572 | 21·502 |
| 10·004 | 712·52 | 0·21 | 0·0007 |
| <u>34·808</u> | | | |

| | | | |
|-------------------|--------|----------|----------|
| (52) | (53) | (54) | (55) |
| 81·0235 | 0·0007 | 8456·5 | 576·34 |
| 576·03 | 5000·0 | 0·37 | 4000·005 |
| 4712·5 | 427·0 | 8456·302 | 213·5 |
| 6·53712 | 37·12 | 0·007 | 2753·0 |
| <u>5376·09062</u> | | | |

MONEY.

| | | | |
|-------------------|----------|------------|-------------|
| (56) | (57) | (58) | (59) |
| £ s. d. | £ s. d. | £ s. d. | £ s. d. |
| 4567 14 6½ | 76 14 7 | 3767 13 11 | 5674 17 6½ |
| 776 15 7½ | 667 13 6 | 4678 14 10 | 4767 16 11½ |
| 76 17 9¼ | 67 15 7 | 767 12 9 | 3466 17 10¾ |
| 51 0 10¼ | 5 4 2 | 10 11 5 | 5984 2 2¼ |
| 44 5 6 | 3 4 | 3 4 11 | 8762 9 9 |
| <u>5516 14 3¼</u> | | | |

AVOIRDUPOIS WEIGHT.

| | | | |
|-----------------|-----------------|-----------------|---------------|
| (60) | (61) | (62) | (63) |
| cwt. qrs. lb. | cwt. qrs. lb. | cwt. qrs. lb. | cwt. qrs. lb. |
| 76 3 14 | 476 1 24½ | 447 1 7 | 14 2 12 |
| 37 2 15 | 756 3 21¼ | 576 1 6 | 3 3 7 |
| 14 1 11 | 767 1 16 | 467 1 7½ | 2 15 |
| | 567 2 15 | 563 1 6 | 7 0 3 |
| <u>128 3 15</u> | <u>973 1 12</u> | <u>428 0 0¼</u> | <u>14</u> |

TROY WEIGHT.

| (64) | (65) | (66) |
|-------------------|-------------------|-------------------|
| lb. oz. dwt. grs. | lb. oz. dwt. grs. | lb. oz. dwt. grs. |
| 7 0 5 9 | 57 9 12 14 | 87 3 7 12 |
| 5 6 6 7 | 67 9 11 11 | 11 12 3 |
| 9 5 6 8 | 66 8 10 5 | 16 14 |
| <u>21 11 18 0</u> | 74 6 5 3 | 44 12 10 13 |
| | <u>12 3 5 4</u> | <u>67 8 9 10</u> |

TIME.

| (67) | (68) | (69) |
|---------------------|-------------------|-------------------|
| yrs. ds. hrs. ms. | yrs. ds. hrs. ms. | yrs. ds. hrs. ms. |
| 99 359 9 56 | 60 90 0 50 | 50 127 7 50 |
| 88 0 8 57 | 6 76 1 57 | 120 9 44 |
| 77 120 7 49 | 3 58 | 76 121 11 44 |
| <u>265 115 2 42</u> | <u>6 1 2 0</u> | 6 47 3 41 |
| | | <u>8 9 11 17</u> |

CLOTH MEASURE.

| (70) | (71) | (72) | (73) |
|----------------|----------------|----------------|----------------|
| yds. qrs. nls. | yds. qrs. nls. | yds. qrs. nls. | yds. qrs. nls. |
| 567 3 2 | 147 3 3 | 157 2 1 | 156 1 1 |
| 476 1 0 | 173 1 0 | 143 3 2 | 176 3 1 |
| 72 3 3 | 148 2 1 | 1 2 | 54 1 0 |
| <u>5 2 1</u> | <u>92 3 2</u> | <u>54 0 3</u> | <u>573 2 3</u> |
| 1122 2 2 | | | |

CANADIAN MONEY.

| (74) | (75) | (76) | (77) |
|------------------|-----------|-----------|-----------|
| \$978.63 | \$ 69.42 | \$719.43 | \$9863.47 |
| 492.29 | 189.87 | 912.99 | 986.10 |
| 83.43 | 674.29 | 68.68 | 91.89 |
| 729.47 | 86.43 | 50.00 | 7.45 |
| 9.00 | 982.78 | 9.73 | .98 |
| <u>\$2292.82</u> | <u>\$</u> | <u>\$</u> | <u>\$</u> |

78. $0.4 + 74.47 + 37.007 + 75.05 + 747.077 = 934.004.$

79. $56.05 + 4.75 + 0.007 + 36.14 + 4.672 = 101.619.$

80. $0.76 + 0.0076 + 76 + 0.5 + 5 + 0.05 = 82.3176.$

81. $0.5 + 0.005 + 5 + 50 + 500 = 555.505.$

82. $0.367 + 56.7 + 762 + 97.6 + 471 = 1387.667.$

83. Add eight hundred and fifty-six thousand, nine hundred and thirty-three; one million, nine hundred and seventy-six thousand, eight hundred and fifty-nine; two hundred and three millions, eight hundred and ninety-five thousand, seven hundred and fifty-two. *Ans.* 206729544.

84. Add three millions, and seventy-one thousand; four millions, and eighty-six thousand; two millions, and fifty-one thousand; one million; twenty-five millions, and six; seventeen millions, and one; ten millions, and two; twelve millions, and twenty-three; four hundred and seventy-two thousand, nine hundred and twenty-three; one hundred and forty-three thousand; one hundred and forty-three millions. *Ans.* 217823955.

85. Add one hundred and thirty-three thousand; seven hundred and seventy thousand; thirty-seven thousand; eight hundred and forty-seven thousand; thirty-three thousand; eight hundred and seventy-six thousand; four hundred and ninety-one thousand. *Ans.* 3187000.

86. Add together one hundred and sixty-seven thousand; three hundred and sixty-seven thousand; nine hundred and six thousand; two hundred and forty-seven thousand; ten thousand; seven hundred thousand; nine hundred and seventy-six thousand; one hundred and ninety-five thousand; ninety-seven thousand. *Ans.* 3665000.

APPLICATIONS.

1. How many miles is it from the lower end of Lake Huron to the Gulf of St. Lawrence, passing through the River St. Clair, 25 miles long; Lake St. Clair, 20 miles; River Detroit, 23 miles; Lake Erie, 250 miles; Niagara River, 34 miles; Lake Ontario, 180 miles; and the River St. Lawrence, 750 miles long?

Ans. 1282 miles.

2. The city of Toronto has a population of about 50000; Hamilton, 25000; Kingston, 15000; London, 10000; Ottawa, 10000; Montreal, 75000; and Quebec, 45000. What is the population of these seven cities taken together?

Ans. 230000.

3. In the year 1856 Canada exported:—Produce of the mine, \$165000; produce of the sea, \$500000; produce of the forest, \$10000000; animals and their produce, \$2500000; agricultural products, \$15000000; manufactures and ships, \$1600000; and various other products to the amount of \$2235000. What was the total value of Canadian exports for that year?

Ans. \$32000000.

4. A wholesale merchant sells, during the year, goods to the amount of \$11080 in Toronto; \$9427 in Galt; \$1798 in Berlin; \$16423 in Hamilton; \$7496 in Guelph; \$6429 in Woodstock; \$5297 in Chatham; and \$8426 in Goderich. Required the amount of the year's sales. *Ans.* \$66376.

5. The Grand Trunk Railway is 962 miles long, and cost \$60000000; the Great Western is 229 miles long, and cost \$14000000; the Ontario, Simcoe, and Huron is 95 miles long, and cost \$3300000; the Toronto and Hamilton is 38 miles long, and cost \$2000000. What is the aggregate length and cost of these four roads? *Ans.* Length, 1324 miles, and cost \$79300000

6. The circulation of promissory notes for the four weeks ending February 3, 1844, was as follows:—Bank of England, about £21228000; private banks of England and Wales, £4980000; Joint Stock Banks of England and Wales, £3446000; all the banks of Scotland, £2791000; Bank of Ireland, £3581000; all the other banks of Ireland, £2429000; what was the total circulation? *Ans.* £38455000.

7. Chronologers have stated that the creation of the world occurred 4004 years before Christ; the deluge, 2348; the call of Abraham, 1921; the departure of the Israelites from Egypt, 1491; the foundation of Solomon's temple, 1012; the end of the captivity, 536. This being the year 1859, how long is it since each of these events?

Ans. From the creation, 5863 years; from the deluge, 4207; from the call of Abraham, 3780; from the departure of the Israelites, 3350; from the foundation of the temple, 2871; and from the end of the captivity, 2395.

8. Add together the following:—2d., about the value of the Roman sestertius; 7½d., that of the denarius; 1½d., a Greek obolus; 9d., a drachma; £3 15s., a mina; £225, a talent; 1s. 7d., the Jewish shekel; and £342 3s. 9d., the Jewish talent.

Ans. £571 2s.

9. Add together 2 dwt. 16 grains, the Greek drachma; 1 lb. 1 oz. 1 dwt., the mina; 67 lb. 7 oz. 5 dwt., the talent.

Ans. 68 lb. 8 oz. 8 dwt. 16 grains.

10. What was the population of the British provinces in North America in 1834, the population of Lower Canada being stated at 549005, of Upper Canada, 336461; of New Brunswick, 152156; of Nova Scotia and Cape Breton, 142548; of Prince Edward's Island, 32292; of Newfoundland, 75000? *Ans.* 1287462.

11. A owes to B £567 16s. 7½d.; to C £47 16s.; and to D £56 0s. 1d. How much does he owe in all? *Ans.* £671 12s 8½d.

12. A man has owing to him the following sums:—£3 10s. 7d.; £46 0s. 7½d.; and 52 14s. 6d. How much is the entire?

Ans. £102 5s. 8½d.

13. A merchant sends off the following quantities of butter:—47 cwt. 2 qrs. 7 lb.; 38 cwt. 3 qrs. 8 lb.; and 16 cwt. 2 qrs. 20 lb. How much did he send off in all? *Ans.* 103 cwt. 10 lb.

14. A merchant receives the following quantities of tallow, viz:—13 cwt. 1 qr. 6 lb.; 10 cwt. 3 qrs. 10 lb.; and 9 cwt. 1 qr. 15 lb. How much has he received in all?

Ans. 33 cwt. 2 qrs. 6 lb.

15. A silversmith has 7 lb. 8 oz. 16 dwts.; 9 lb. 7 oz. 3 dwts.; and 4 lb. 1 dwt. What quantity has he?

Ans. 21 lb. 4 oz.

16. A merchant sells to A, 76 yards 3 quarters 2 nails; to B, 90 yards 3 quarters 3 nails; and to C, 190 yards 1 nail. How much has he sold in all?

Ans. 357 yards 3 quarters 2 nails.

17. A merchant in Toronto sells goods to the following amounts during the week, viz:—Monday, \$429.38; Tuesday, \$711.43; Wednesday, \$419.87; Thursday, \$1080.42; Friday, \$1304.65; Saturday, \$2498.91. Required the whole amount of the week's sales.

Ans. \$6444.66.

18. Looking over my last month's expenditure, I find that I have paid the following sums, viz.—Baker's bill, \$5.73; Butcher's bill, \$20.91; Groceries, \$12.75; Fruit, \$3.29; Rent, \$16.25; Servants' wages, \$10; Tailor's accounts, \$17.87; Shoemaker's bill, \$11.63; and sundries, \$9.47. Required how much I paid in all.

Ans. \$107.90.

19. Add together \$607.19; \$298.97; \$789.87; \$1723.10; and \$123.00.

Ans. \$3542.13.

20. A farmer sells seven loads of wheat, the first containing 1763 lbs., the second 1827 lbs., the third 1329 lbs., the fourth 1901 lbs., the fifth 1666 lbs., the sixth 1879 lbs., and the seventh 1185 lbs. What was the aggregate weight of the seven loads and how many bushels did they contain?

Ans. 11550 lbs. or 192½ bushels.

NOTE.—The bushels are found by dividing the aggregate weight by 60 lbs., the weight of one bushel.

21. Having effected an insurance on my household furniture, &c., I am required to make a detailed statement of its value. I find this to be as follows:—Carpets \$350.00, table and bed linen \$90.88, beds and bedding \$173.60, furniture \$791.23, pictures and engravings \$207.18, books \$1649.19, plate and plated ware \$307.18. Required the total value of my household furniture.

Ans. \$3469.26.

22. Toronto has a population of 45000, Hamilton 20000, Brockville 4000, Prescott 2500, Kingston 15000, Ottawa City 10000, Chatham 4000, Goderich 2000, London 10000, Port Hope 4000, Cobourg 5000, Montreal 70000, and Quebec 50000. What is the entire population of these 13 cities and towns?

Ans. 241500.

20. The pupil should not be allowed to leave addition until he can read up the columns without hesitation. For instance, in the following questions, which are inserted for the sake of practice in rapid addition, he should not be permitted to spell the columns thus, 6 and 4 are 10, and 4 are 14, and 4 are 18, and 5 are 23

&c., but should be required to *read* them, *i. e.*, simply touch each digit with his pencil and name the sum, thus:—6, 10, 14, 18, 23, 31, 32, 35, 42, 43, 44, 49, 53, &c. &c.

| I. | II. | III. | IV. |
|--------|--------|--------|--------|
| 244658 | 275634 | 135790 | 123456 |
| 492327 | 386731 | 246824 | 786123 |
| 635425 | 987654 | 135790 | 456789 |
| 321465 | 321456 | 864212 | 123456 |
| 732849 | 989123 | 579246 | 788123 |
| 376731 | 456789 | 835792 | 459789 |
| 935746 | 123456 | 468357 | 123456 |
| 847963 | 789123 | 924689 | 789123 |
| 745143 | 456789 | 753246 | 456789 |
| 234561 | 123456 | 835792 | 123456 |
| 746874 | 789123 | 468357 | 789123 |
| 934746 | 456789 | 924683 | 456789 |
| 872345 | 123459 | 579246 | 123456 |
| 934756 | 789123 | 835798 | 789123 |
| 842345 | 456789 | 642875 | 456789 |
| 873456 | 123456 | 334683 | 123456 |
| 864580 | 789123 | 579864 | 789122 |
| 234672 | 456789 | 297531 | 456789 |
| 325871 | 246842 | 135795 | 871178 |
| 479234 | 357931 | 246834 | 936639 |
| 845645 | 642248 | 824248 | 248842 |
| 823456 | 756139 | 357964 | 525255 |
| 245734 | 246842 | 872278 | 736376 |
| 872475 | 657931 | 375946 | 875578 |
| 896731 | 642248 | 624862 | 473468 |
| 456841 | 753139 | 375937 | 934579 |
| 314567 | 246842 | 872459 | 894645 |
| 814563 | 357931 | 837645 | 123875 |
| 427831 | 642248 | 644875 | 767457 |
| 932768 | 753913 | 472963 | 875345 |
| 456345 | 375913 | 875847 | 874563 |
| 345634 | 426428 | 864314 | 375534 |
| 734734 | 573931 | 734561 | 937565 |
| 734564 | 624824 | 273475 | 875734 |
| 834756 | 735813 | 845675 | 698945 |

RECAPITULATION.

I. Addition is the process of finding the sum of two or more numbers.

II. The numbers to be added are called *Addends*.

III. The *result* of the addition is called the *sum* of the addends.

IV. In writing numbers down preparatory to adding them, we write units under units, tens under tens, &c., because it is more convenient, since only *like* quantities, *i. e.*, quantities of the same name can be added together.

V. We draw a line under the addends in order to separate them from the *sum*.

VI. We begin the addition at the column containing the lowest denomination and work from right to left, because, by so doing, we are enabled to *carry*, from the column added, the number of units of the next higher denomination it contains, to their appropriate column, and thus perform the work by one addition which would otherwise require two or more.

VII. We divide the sum of the units of any one denomination by the number required to make one of the next higher, in order to know how many we are to carry to the next higher.

VIII. The addition of simple numbers was formerly called Simple Addition; and the addition of compound, or denominate numbers, Compound Addition. As the same rule applies to the addition of all numbers, there is no reason why, in a second course, we should treat of the addition of simple and denominate numbers separately.

QUESTIONS.

NOTE.—*Arabic numerals, thus (14), refer to the articles of the Section and Roman numerals, thus (VI.), to the Recapitulation.*

1. Into what parts may Arithmetic be divided? (I.)
2. Of what does the Arithmetic of Whole Numbers treat? (I.)
3. What rules are included in the Arithmetic of Whole Numbers? (2)
4. Of what does the Arithmetic of Fractions treat? (1)
5. How is the Arithmetic of Fractions divided? (3)
6. How is the unit divided in Vulgar or Common Fractions? (3)
7. How is the unit divided in Decimal Fractions? (3)
8. Of what does the Arithmetic of Ratios treat? (1)
9. What rules of Arithmetic are embraced in the Arithmetic of Ratios? (4)
10. What are the fundamental rules of Arithmetic? (5)
11. Why are they so called? (5)
12. Upon what rules do all the operations of Arithmetic ultimately depend? (6)
13. What is the *sum* of two numbers? (7)
14. What is Addition? (8 or I.)
15. What are addends? (9 or II.)
16. What kind of quantities only can be added? (10)
17. What is the rule for Addition? (15)
18. Why must we place units of the same denomination in the same vertical column? (IV.)

19. Why do we draw a line under the addends ? (V.)
20. Why do we begin to add at the lowest denominations ? (VI.)
21. Why do we divide the sum of the units of any one denomination by as many as make one of the next higher ? (VII.)
22. How do we prove addition ? (19)
23. Upon what axiom is the 2nd method of proof founded ? (19)
24. So far as the result is concerned, does it make any difference where we commence to add ? (12)
25. Exhibit the work when we commence adding at the left-hand side, or highest denomination. (12)
26. When the addends are very numerous, what plans may we adopt ? (18)
27. Upon what principle does the former of these plans proceed ? (18)
28. What different rules were formerly made in addition ? (VIII.)
29. Is this distinction necessary ? Why not ? (VIII.)
30. Illustrate the difference between *spelling* and *reading* in addition. (20)

SUBTRACTION.

21. Subtraction is the process of finding the difference between two numbers.

22. The greater of the two given numbers, or that which is *to be lessened*, is called the *Minuend* (Lat. *Minuendus* "to be lessened"); the smaller, or that which is *to be subtracted*, the *Subtrahend* (Lat. *Subtrahendus*, "to be subtracted").

23. If anything is left after making the subtraction, it is called the *remainder*, *difference*, or *excess*.

24. Only quantities of the same denomination (i. e. which have the same unit) can be subtracted the one from the other.

25. Subtraction is indicated by —, called the minus, or negative sign. Thus $5-4=1$, read five minus four equal to one, indicates that if 4 is subtracted from 5, unity is left.

Quantities connected by the negative sign cannot be taken, indifferently, in any order; because, for example, $5-4$ is not the same as $4-5$. In the former case the positive quantity is the greater, and 1 (which means + 1) is left; in the latter, the negative quantity is the greater, and —1, or one to be subtracted, still remains. To illustrate yet further the use and nature of the signs, let us suppose that we *have* five pounds and *owe* four;—the five pounds we *have* will be represented by 5, and our debt by —4; taking the 4 from the 5, we shall have 1 pound (+1) remaining. Next let us suppose that we *have* only four pounds and *owe* five; if we take the 5 from the 4 (that is, if we *pay* as far as we can) a debt of one pound represented by —1, will still remain; consequently $5-4=1$; but $4-5=-1$.

26. When several numbers, connected by the signs + and — are placed within brackets, thus, $(7+4-6-3+9)$, the whole expression is to be considered as one quantity. The negative sign before such an expression indicates that the *value* of the whole expression within the brackets, is to be subtracted, or, what amounts to the same thing, that the numbers having the sign + before them are to be subtracted, and those having the sign —, added. Hence a minus sign before a bracket, has the effect of changing the signs of all the quantities within the brackets, when the brackets are removed. So, also, when we desire to place a quantity within brackets, we must change its sign, if the sign preceding the first bracket be minus.

The following examples will show how the brackets affect numbers, according as we make them include an additive, or a subtractive quantity:—

$$27-4+7-3 = 27$$

$$27-(4+7-3) = 19$$

But $27-(4-7+3) = 27$. [changing all the signs of the original quantities, but the first.]

Again $48+7-3-8+7-2 = 49$.

$$48+(7-3-8+7-2) = 49; \text{ what is in the brackets being additive, it is not necessary to change any signs.}$$

$$48+7-(3+8-7+2) = 49; \text{ It is now necessary to change all the signs in the brackets.}$$

$$48+7-3-(8-7+2) = 49; \text{ it is necessary in this case, also, to change the signs.}$$

$$48+7-3-8+(7-2) = 49; \text{ it is not necessary in this case.}$$

27. When the numbers are small they can be subtracted mentally, thus: from 6 shillings take 4 shillings, and the result is evidently 2 shillings; from 9 pounds take 4 pounds, and the remainder is 5 pounds; from 16 days take 9 days, and the remainder is 7 days; from 14 sixteenths take 5 sixteenths, and the remainder is 9 sixteenths, &c.

When the numbers are too large to be conveniently retained in the mind, they may be written as in addition.

EXAMPLE 1.—From 97 take 43, that is, from 9 tens and 7 units take 4 tens and 3 units.

OPERATION.

$$90+7 \text{ or } 97 = \text{Minuend.}$$

$$40+3 \text{ or } 43 = \text{Subtrahend.}$$

$$\begin{array}{r} 90+7 \\ -40+3 \\ \hline 50+4 \end{array} \quad 54 = \text{Remainder.}$$

EXPLANATION.—3 units from 7 units leaves 4 units, and 40 units or 4 tens from 90 units or 9 tens, leave 50 units or 5 tens.

EXAMPLE 2.—Let it be required to subtract 746 from 978, or from $900+70+8$ to take $700+40+6$.

| | | | |
|---------------|------|-------|--------|
| OPERATION. | hun. | tens. | units. |
| $900+70+8$ or | 9 | 7 | 8 |
| $700+40+6$ or | 7 | 4 | 6 |
| $200+30+2$ or | 2 | 3 | 2 |

EXPLANATION.—6 units from 8 units, and 2 units remain; 40 units or 4 tens from 70 units or 7 tens, and 30 units or 3 tens remain; and 700 units or 7 hundreds, from 900 units or 9 hundreds, and 200 units, or 2 hundreds remain.

EXAMPLE 3.—From 842 take 661.

EXPLANATION.—In placing the subtrahend under the minuend, in this example, we find that, while we can subtract the units from the units, we cannot subtract the tens from the tens, since we have 6 tens in the subtrahend and only 4 tens in the minuend. We get over this difficulty by considering the minuend to be, not $800+40+2$, but $700+140+2$, or, in other words, we *borrow* one of the order of hundreds and reduce it to tens. Now we have 1 unit from 2 units and 1 unit remains; 60 units or 6 tens from 140 units or 14 tens, and 80 units or 8 tens remain; 600 units or 6 hundreds, from 700 units or 7 hundreds, and 100 units or 1 hundred remain.

EXAMPLE 4.—Let it be required to subtract 3 cwt. 2 qrs. 7 lbs. from 9 cwt. 1 qr. 8 lbs.

EXPLANATION.—As we cannot subtract 2 qrs. from 1 qr. we *borrow* 1 cwt. and reduce it to quarters. The 9 cwt. cwt. qrs. lb. 1 qr. 8 lb. we then consider as 8 cwt. 5 qrs. 8 lb. and from it subtract the 5 cwt. 3 qrs. 7 lb. Thus, 7 lbs. from 8 lbs. and 1 lb. remains; 2 qrs. from 5 qrs. and 3 qrs. remain; and 3 cwt. from 8 cwt. and 5 cwt. remain.

28. Hence, to find the difference between two numbers, we deduce the following:—

RULE.

Write the subtrahend under the minuend, so that units of the same denomination may be in the same vertical column. (24) Draw a line under the subtrahend to separate it from the remainder. Subtract each digit in the subtrahend from the one over it in the minuend, beginning at the lowest denomination.

When the units of any one denomination of the minuend fall short of those of the same denomination in the subtrahend, borrow one of the next higher denomination in the minuend, reduce it to its equivalent units of the required denomination, add them to the units of that denomination given in the minuend, and from their sum subtract the units of that denomination given in the subtrahend.

29. The following is the complete work of a question in Subtraction:

EXAMPLE 5.—From 6400 lbs. 0 oz. 0 dwt. 7·0006 grs. take 987 lbs, 3 oz. 17 dwt. 22·6349 grs.

| | | | | OPERATION. | | |
|--------------|-------|--------|-----------------|-------------|--|--|
| (10) 9 9 | 11 | 19 | 24·9 9 9 | | | |
| 5 3 10 10 | 12 | 20 | 6·10 10 10 (10) | | | |
| 6 4 0 0 lbs. | 0 oz. | 0 dwt. | 7·0 0 0 6 grs. | Minuend. | | |
| 9 8 7 | 3 | 17 | 22·6 3 4 9 | Subtrahend. | | |
| <hr/> | | | | | | |
| 5 4 1 2 | 8 | 2 | 8·3 6 5 7. | Remainder. | | |

EXPLANATION.—Here, as we cannot take 9 tenths of thousandths of a grain from 6 tenths of thousandths of a grain, we borrow one grain, there being no tenths, hundredths, or thousandths in the minuend. Now this one grain is equivalent to ten of the order of tenths of grains. Borrow one tenth and there remain 9 tenths, and the one tenth we borrowed is equal to 10 hundredths. Borrow 1 hundredth, there remain 9 hundredths, and the one hundredth we borrowed is equal to 10 thousandths. Borrow 1 thousandth, there remain 9, and the 1 thousandth is equal to 10 of the order of tenths of thousandths—the order for which it was necessary to borrow. 10 of the order of tenths of thousandths of grains and 6 of the order of tenths of thousandths of grains, make 16, from which take 9 of the order of tenths of thousandths of grains, and there remain 7 of the order of tenths of thousandths of grains; 4 of the order of thousandths from 9 of the order of thousandths and 5 of the order of thousandths remain; 3 of the order of hundredths from 9 of the order of hundredths and 6 hundredths remain: 6 tenths from 9 tenths and 3 tenths remain.

Again, as we cannot take 22 grains from 6 grains, we borrow from the next available higher order, which, in this case, is hundreds of pounds. 1 of the order of hundreds of pounds reduced, as above, to its equivalent lower denomination, is equal to 9 tens of lbs., 9 units of lbs. 11 oz. 19 dwt. 24 grs. 2½ grains, added to 6, make 30 grains, and 22 grains from 30 grains, leave 8 grains; 17 dwt. from 19 dwt. leave 2 dwt.; 3 oz. from 11 oz. leave 8 oz.; 7 units of lbs. from 9 units of lbs. leave 2 units of lbs.; 8 tens of lbs. from 9 tens of lbs. leave 1 ten of lbs. We cannot take 9 hundreds of lbs., from 3 hundreds of lbs., so we are compelled to borrow 1 of the order of thousands of lbs., which is equal to 10 hundreds of lbs., and 3 hundreds of lbs., make 13 hundreds of lbs.; 9 hundreds of lbs. from 13 hundreds of lbs. and 4 hundreds of lbs. remain; 0 thousands of lbs. from 5 thousands of lbs. and 5 thousands of lbs. remain.

30. If any digit of the minuend be smaller than the corresponding digit of the subtrahend, practically, we can proceed in either of two ways. First, we may increase that denomination of the minuend which is too small, by borrowing *one* from the next higher, (considered as so many of the lower denomination, or that which is to be increased,) and adding it to those of the lower, already in the minuend. In this case we alter the form, but not the value of the minuend; which, in the example given below, would become—

| hundreds. | tens. | units. | |
|-----------|-------|--------|------------------------|
| 7 | 8 | 12 | = 792, the minuend. |
| 4 | 2 | 7 | = 427, the subtrahend. |
| 3 | 6 | 5 | = 365, the difference. |

Or, secondly, we may add equal quantities to both minuend and subtrahend, which will not alter the difference; then we would have

| hundreds. | tens. | units. | |
|-----------|-------|--------|----------------------------------|
| 7 | 9 | 2 + 10 | = 792 + 10, the minuend + 10. |
| 4 | 2 + 1 | 7 | = 427 + 10, the subtrahend + 10. |
| 3 | 6 | 5 | = 365 + 0, the same difference. |

In this mode of proceeding we do not use the *given* minuend and subtrahend, but others which produce the same remainder.

PROOF OF SUBTRACTION.

31. **FIRST METHOD.**—*Add together the remainder and subtrahend; the sum should be equal to the minuend.*

For the remainder expresses by how much the subtrahend is smaller than the minuend; adding, therefore, the remainder to the subtrahend, should make it equal to the minuend; thus,

| | | |
|------|-------------|---|
| 8754 | minuend. | } |
| 5839 | subtrahend. | |
| 2915 | difference. | |

Sum of difference and subtrahend, 8754 = minuend.

SECOND METHOD.—*Subtract the remainder from the minuend, and what is left should be equal to the subtrahend.*

For the remainder is the excess of the minuend over the subtrahend; therefore, taking away this excess should leave both equal; thus

| | |
|------------------|-----------------------------------|
| 8634 minuend. | PROOF: 8634 minuend. |
| 7985 subtrahend. | 649 remainder. |
| 649 remainder. | New remainder, 7985 = subtrahend. |

In practice, it is sufficient to set down the quantities once; thus

| |
|------------------|
| 8634 minuend. |
| 7985 subtrahend. |
| 649 remainder. |

Difference between remainder and minuend, 7985 = subtrahend.

EXERCISE.

| | (6) | (7) | (8) | (9) | (10) |
|------|----------|---------|---------|---------|---------|
| From | 11000000 | 3000001 | 8000800 | 8000000 | 4040053 |
| Take | 9919919 | 2199077 | 377776 | 62358 | 220202 |
| | 1080081 | | | | |

| | (11) | (12) | (13) | (14) | (15) |
|------|-------|-------|---------|--------|--------|
| From | 85.73 | 865.5 | 594.763 | 47.630 | 52.137 |
| Take | 42.16 | 73.2 | 85.6 | 0.078 | 20.005 |
| | 43.57 | | | | |

| | (16) | (17) | (18) | (19) | (20) |
|------|---------|---------|--------|----------|----------|
| From | 0.00063 | 874.32 | 57.004 | 47632.0 | 400.3270 |
| Take | 0.00048 | 5.63705 | 2.3 | 0.845003 | 0.006 |
| | 0.00015 | | | | |

| | |
|---------------------------------|---------------------------------|
| 21. 7465676 — 567456 = 6898220. | 32. 97777 — 4 = 97773. |
| 22. 566789 — 75674 = 491115. | 33. 60000 — 1 = 59999. |
| 23. 941000 — 5007 = 935993. | 34. 75477 — 76 = 75401. |
| 24. 97001 — 50077 = 46924. | 35. 7.97 — 1.05 = 6.92. |
| 25. 76734 — 977 = 75757. | 36. 1.75 — 0.074 = 1.676. |
| 26. 56400 — 100 = 56300. | 37. 97.07 — 4.769 = 92.301. |
| 27. 700000 — 99 = 699901. | 38. 7.05 — 4.776 = 2.274. |
| 28. 5700 — 500 = 5200. | 39. 10.761 — 9.001 = 1.76. |
| 29. 9777 — 89 = 9688. | 40. 12.10009 — 7.121 = 4.97909. |
| 30. 76000 — 1 = 75999. | 41. 176.1 — 0.007 = 176.093. |
| 31. 90017 — 3 = 90014. | 42. 15.06 — 7.863 = 7.197. |

MONEY.

| | | | | |
|------|---------------|---------------|--------------|-------------|
| | (43) | (44) | (45) | (46) |
| From | \$9876.43 | \$427.63 | \$721.73 | \$16.25 |
| Take | <u>987.49</u> | <u>197.21</u> | <u>91.00</u> | <u>9.75</u> |
| | \$8888.94 | \$230.42 | \$ | \$ |

| | | | | |
|------|---------------|--------------|---------------|-------------|
| | (47) | (48) | (49) | (50) |
| From | \$1234.50 | \$671.98 | \$236.29 | \$7.19 |
| Take | <u>999.96</u> | <u>99.67</u> | <u>611.99</u> | <u>1.86</u> |
| | \$234.54 | \$572.31 | \$ | \$ |

| | | | | | |
|------|-----------------|-----------------|-------------|----------------|---------------|
| | (51) | (52) | (53) | (54) | (55) |
| | £ s. d. | £ s. d. | £ s. d. | £ s. d. | £ s. d. |
| From | 1098 12 6 | 767 14 8 | 76 15 6 | 47 16 7 | 97 14 6 |
| Take | <u>434 15 8</u> | <u>486 13 9</u> | <u>14 5</u> | <u>39 17 4</u> | <u>6 15 7</u> |
| | £663 16 10 | | | | |

| | | | | | |
|------|----------------|----------------|----------------|-----------------|-----------------|
| | (56) | (57) | (58) | (59) | (60) |
| | £ s. d. | £ s. d. | £ s. d. | £ s. d. | £ s. d. |
| From | 98 14 2 | 47 14 6 | 97 16 6 | 147 14 4 | 560 15 6 |
| Take | <u>77 15 3</u> | <u>38 19 9</u> | <u>88 17 7</u> | <u>120 10 8</u> | <u>477 17 7</u> |

AVOIRDUPOIS WEIGHT.

| | | | | |
|------|----------------|---------------|------------------|----------------|
| | (61) | (62) | (63) | (64) |
| | cwt. qrs. lb. | cwt. qrs. lb. | cwt. qrs. lb. | cwt. qrs. lb. |
| From | 200 2 24 | 175 2 15 | 9664 2 23 | 554 0 0 |
| Take | <u>99 3 15</u> | <u>27 2 7</u> | <u>9073 0 24</u> | <u>476 3 5</u> |
| | 100 3 9 | | | |

TROY WEIGHT.

| | | | |
|------|-------------------|-------------------|--------------------|
| | (65) | (66) | (67) |
| | lb. oz. dwt. grs. | lb. oz. dwt. grs. | lb. oz. dwt. grs. |
| From | 554 9 19 4 | 946 0 10 0 | 917 0 14 9 |
| Take | <u>97 0 16 15</u> | <u>17 23</u> | <u>798 0 18 17</u> |
| | 457 9 2 13 | | |

TIME.

| | (68) | | | | (69) | | | | (70) | | | |
|------|-------|-----|------|-----|-------|-----|------|-----|-------|-----|------|-----|
| | yrs. | ds. | hrs. | ms. | yrs. | ds. | hrs. | ms. | yrs. | ds. | hrs. | ms. |
| From | 767 | 131 | 6 | 30 | 475 | 14 | 13 | 16 | 567 | 126 | 14 | 12 |
| Take | 476 | 110 | 14 | 13 | 160 | 16 | 13 | 17 | 400 | 0 | 15 | 0 |
| | <hr/> | | | | <hr/> | | | | <hr/> | | | |
| | 291 | 20 | 16 | 17 | | | | | | | | |

APPLICATIONS.

1. A shopkeeper bought a piece of cloth containing 42 yards for £22 10s., of which he sells 27 yards for £15 15s.; how many yards has he left, and what have they cost him?

Ans. 15 yards; and they cost him £6 15s.

2. A merchant bought 234 tons, 17 cwt., 1 quarter, 23 lb., and sold 147 tons, 18 cwt., 2 quarters, 24 lb.; how much remained unsold?

Ans. 86 tons, 18 cwt. 2 qrs. 24 lb.

3. In 1856 the revenue of Canada was as follows:—customs, \$4500000; public works, \$500000; crown lands, \$500000; and casual, \$320000. For the same year the expenditure was as follows:—interest on public debt, &c., \$1000000; civil government, \$225000; legislation, \$450000; administration of justice, \$450000; education, \$380000; collection of revenue, \$940000; public works, &c., \$1755000. How much did the total revenue of that year exceed the total expenditure?

Ans. \$620000.

4. The census of 1852 gives the population of Upper Canada as 962004, and that of Lower Canada as 890261. By how much did the population of the former exceed that of the latter?

Ans. 71743.

5. Upper Canada contains 147832 square miles; Lower Canada, 209990 square miles; Nova Scotia and Cape Breton, 18746 square miles; New Brunswick, 27620 square miles; Prince Edward's Island, 2173 square miles; Newfoundland, 36000 square miles; and Hudson's Bay Territory, 2436000 square miles. By how much does the aggregate extent of these British North American Provinces fall short of the total area of the United States—the latter being 2936116 square miles?

Ans. 57755 square miles.

6. A merchant has 209 casks of butter, weighing 400 cwt. 2 qrs. 14 lb.; and ships off 173 casks, weighing 213 cwt. 2 qrs. 24 lb. How many casks has he left; and what is their weight?

Ans. 36 casks, weighing 186 cwt. 3 qrs. 15 lb.

7. If from a piece of cloth containing 496 yards, 3 quarters, and 3 nails, 1 cut 247 yards, 2 quarters, 2 nails, what is the length of the remainder? *Ans.* 249 yards, 1 quarter, 1 nail.

8. A field contains 769 acres, 3 roods, and 20 perches, of

which 576 acres, 2 roods, 23 perches are tilled; how much remains untilled? *Ans.* 193 acres, 37 perches.

9. I owed my friend a bill of £76 16s. 9½d., out of which I paid £59 17s. 10½d.; how much remained due?

Ans. £16 18s. 10¾d.

10. The population of London is 2363141, and that of Paris is 1053262. How much does the population of London exceed that of Paris?

Ans. 1309879.

11. The population of Liverpool is 384265, and that of New York 515547. How much does the population of New York exceed that of Liverpool?

Ans. 131282.

12. Lake Huron contains 20000 square miles: by how much does it exceed the area of Lakes Erie and Ontario—the former containing 11000 square miles, and the latter 7000 sq. miles?

Ans. 2000 square miles.

13. A merchant has \$6947.87 in bank; \$4789.63 in stock; \$9491.11 in property; and \$14167.93 on his books against his customers: his debts amount to \$19476.25. How much is he worth after paying what he owes?

Ans. \$15918.29.

14. What is the value of $6-3+15-4$?

Ans. 14.

15. Of $43+(7-3-14)$?

Ans. 33.

16. Of $47.6-(2+1-24+16-0.34)$?

Ans. 52.94.

17. What is the difference between $15+13-6-81$ and $15+13-(6-81+62)$?

Ans. 100.

32. Before the pupil leaves subtraction he should be able to take any of the nine digits, continually, from a given number, without stopping or hesitating, thus, in subtracting 7 continually from 94, he should say, 94, 87, 80, 73, 64, 57, &c. In the following examples, which are inserted for practice, he should not be allowed to spell the subtraction, thus, 6 from 9 and 3 remain, 4 from 2, we can't, but 4 from 12 and 8 remain, &c.; but should be required to read as follows:—6, 9..3; 4, 12..8; 8, 13..5; 9, 11..2; 9, 18..9, &c.

(18)

9800046043019181697800041081329
 191347813191681473199916199846

(19)

74321913047123098706540456007139
 1342345678912345678912345678912

RECAPITULATION.

I. Subtraction is the process of finding the difference between two numbers.

II. The greater of the two numbers is called the *minuend*.

III. The smaller of the two numbers is called the *subtrahend*.

IV. What is left after making the subtraction is called the *remainder* or difference.

V. Only quantities of the same denomination can be subtracted.

VI. Subtraction is indicated by the sign—which is called minus, or the negative sign.

VII. When several numbers are inclosed in brackets, they are to be considered as constituting only one quantity.

VIII. When a negative sign precedes the first bracket it indicates that all the quantities within the brackets are to have their signs changed when the brackets are removed.

IX. When quantities are removed *into* brackets, preceded by the negative sign, all their signs must be changed.

X. We begin subtraction at the lowest denomination, because it is sometimes necessary to borrow from the higher denominations and reduce.

XI. Instead of thus borrowing and reducing, we may consider any denomination in the minuend increased by as many units of that denomination as make one of the next higher, and then add one to the next higher denomination in the subtrahend. This is merely adding the same quantity under different forms to both minuend and subtrahend, and consequently cannot affect the value of the remainder. (31)

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—Numbers in Roman numerals, thus (V), refer to the *Recapitulation*; those in Arabic numerals, thus (25), refer to the articles of the *Section*.

1. What is Subtraction? (I)
2. What is the minuend? (II.)
3. What is the derivation of the word *minuend*? (22)
4. What is the subtrahend? (III.)
5. What is the derivation of the word *subtrahend*? (22)
6. What is the remainder? (IV.)
7. What kind of quantities can be subtracted? (V.)
8. How is subtraction indicated? (VI.)
9. When several numbers are inclosed together in brackets, how are they to be taken? (VIII and 26.)
10. What effect has a negative sign preceding brackets? (VIII. and 27.)
11. When quantities are removed into brackets, preceded by the sign—what must be done with them? (IX and 27.)
12. What is the rule for subtraction? (28)

13. Why must we put units of the same denomination in the same vertical column? (24)
14. When a digit in the subtrahend is greater than the corresponding digit in the minuend, what is done? (27 Example 3, or 29)
15. What other plan may be adopted? (30)
16. Upon what principle does this plan proceed? (XI.)
17. Why do we begin to subtract at the right-hand side? (X.)
18. How do we prove subtraction? (31)
19. Upon what principles are these methods of proof founded? (31)
20. Illustrate the difference between *spelling* and *reading* in subtraction. (32)

MULTIPLICATION.

33. Multiplication is a short process of taking one number as many times as there are units in another. Hence multiplication is a short method of performing addition.

34. The number to be taken or multiplied is called the *multiplicand*, and in addition would be called an *addend*.

35. The number denoting how many times the multiplicand is to be taken, or, in other words, that by which we multiply, is called the *multiplier*.

36. The number arising from taking the multiplicand as many times as there are units in the multiplier, is called the *product*, and corresponds to the *sum of the addends* in addition.

The multiplicand and multiplier are called the *factors* of the product because they *make* or *produce* it, (Lat. *factor*, "a maker, agent, or producer.")

37. A prime number is one which cannot be exactly divided by any *whole* number, except the unit *one* and *itself*.

38. A composite number is the product of two or more integral factors, *neither* of which is unity. Thus 16 is a composite number, and its factors are 8 and 2, or 4 and 4.

39. Since the product is the result which arises from taking the multiplicand as many times as there are units in the multiplier, it follows;

1st. If the multiplier be equal to unity, the product will be equal to the multiplicand.

2nd. If the multiplier be greater than unity, the product will be as many times greater than the multiplicand as the multiplier is greater than unity.

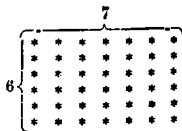
3rd. If the multiplier be less than unity, that is, if it be

a proper fraction, the product will be as many times *less* than the multiplicand as the multiplier is less than unity.

40. Let it be required to multiply any two numbers together, say 7 and 6.

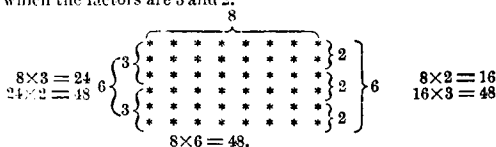
If we make in a horizontal line as many stars as there are units in the multiplicand, and make as many such lines of stars as there are units in the multiplier, it is manifest that the entire number of stars will represent the number of units which result from taking the multiplicand as many times as there are units in the multiplier.

But it is evident that we may consider the 42 stars in the above figure, either as 7 stars taken 6 times, or as 6 stars taken 7 times, that is, $6 \times 7 = 42 = 7 \times 6$.



Hence either of the factors may be used as multiplier without altering the product.

41. Let it be required to multiply the number 8 by the composite number 6, of which the factors are 3 and 2.



If we write 8 stars in a horizontal line and make 6 such lines, we shall evidently have in all $8 \times 6 = 48$, the number of units in all the lines.

But we may consider the 6 lines as 2 sets of 3 lines each, and in each set of 3 lines there are $8 \times 3 = 24$ units. Therefore in the 2 sets there are $24 \times 2 = 48$ units. Again we may consider the 6 lines as 3 sets of 2 lines each, and in each set of 2 lines there are $8 \times 2 = 16$ units. Therefore in 3 such sets there are $16 \times 3 = 48$ units.

Hence $8 \times 6 = 48$

$$8 \times 3 = 24 \text{ and } 24 \times 2 = 48 = 8 \times 6$$

$$8 \times 2 = 16 \text{ and } 16 \times 3 = 48 = 8 \times 6$$

And as the same may be shewn for any other composite number as well for 6, we may conclude that,

When the multiplier is a composite number we may multiply by each of the factors in succession, and the last product will be the entire product sought.

42. As the multiplication of the higher numbers may be resolved into the multiplication of one digit by another, the pupil should make himself perfectly familiar with the following table :

This table is called the Multiplication Table, and was calculated by Pythagoras, a celebrated Greek philosopher who flourished about 500 years before Christ. It was calculated after the following manner:—2 and 2 are 4—twice 2 are 4; 3 and 3 are 6; twice 3 are 6; 4 and 4 are 8—twice 4 are 8, &c.

MULTIPLICATION TABLE.

| | | | | | |
|---------|----------|----------|----------|----------|---------|
| Twice | 3 times | 4 times | 5 times | 6 times | 7 times |
| 1 are 2 | 1 are 3 | 1 are 4 | 1 are 5 | 1 are 6 | 1 are 7 |
| 2 — 4 | 2 — 6 | 2 — 8 | 2 — 10 | 2 — 12 | 2 — 14 |
| 3 — 6 | 3 — 9 | 3 — 12 | 3 — 15 | 3 — 18 | 3 — 21 |
| 4 — 8 | 4 — 12 | 4 — 16 | 4 — 20 | 4 — 24 | 4 — 28 |
| 5 — 10 | 5 — 15 | 5 — 20 | 5 — 25 | 5 — 30 | 5 — 35 |
| 6 — 12 | 6 — 18 | 6 — 24 | 6 — 30 | 6 — 36 | 6 — 42 |
| 7 — 14 | 7 — 21 | 7 — 28 | 7 — 35 | 7 — 42 | 7 — 49 |
| 8 — 16 | 8 — 24 | 8 — 32 | 8 — 40 | 8 — 48 | 8 — 56 |
| 9 — 18 | 9 — 27 | 9 — 36 | 9 — 45 | 9 — 54 | 9 — 63 |
| 10 — 20 | 10 — 30 | 10 — 40 | 10 — 50 | 10 — 60 | 10 — 70 |
| 11 — 22 | 11 — 33 | 11 — 44 | 11 — 55 | 11 — 66 | 11 — 77 |
| 12 — 24 | 12 — 36 | 12 — 48 | 12 — 60 | 12 — 72 | 12 — 84 |
| 8 times | 9 times | 10 times | 11 times | 12 times | |
| 1 are 8 | 1 are 9 | 1 are 10 | 1 are 11 | 1 are 12 | |
| 2 — 16 | 2 — 18 | 2 — 20 | 2 — 22 | 2 — 24 | |
| 3 — 24 | 3 — 27 | 3 — 30 | 3 — 33 | 3 — 36 | |
| 4 — 32 | 4 — 36 | 4 — 40 | 4 — 44 | 4 — 48 | |
| 5 — 40 | 5 — 45 | 5 — 50 | 5 — 55 | 5 — 60 | |
| 6 — 48 | 6 — 54 | 6 — 60 | 6 — 66 | 6 — 72 | |
| 7 — 56 | 7 — 63 | 7 — 70 | 7 — 77 | 7 — 84 | |
| 8 — 64 | 8 — 72 | 8 — 80 | 8 — 88 | 8 — 96 | |
| 9 — 72 | 9 — 81 | 9 — 90 | 9 — 99 | 9 — 108 | |
| 10 — 80 | 10 — 90 | 10 — 100 | 10 — 110 | 10 — 120 | |
| 11 — 88 | 11 — 99 | 11 — 110 | 11 — 121 | 11 — 132 | |
| 12 — 96 | 12 — 108 | 12 — 120 | 12 — 132 | 12 — 144 | |

It appears from this table, that the multiplication of the same two numbers in whatever order taken, produces the same product.

NOTE.—Though the part of the multiplication table given above is enough for the pupil to commit to memory at first; yet, after he has made some proficiency in arithmetic, he may find it advantageous to commit what follows, as it will enable him, in many cases, to shorten his work in a considerable degree. The labour of committing a still more extended table would be scarcely compensated by the advantage resulting.

| | | | | | | |
|----------|----------|----------|----------|----------|----------|----------|
| 13 times | 14 times | 15 times | 16 times | 17 times | 18 times | 19 times |
| 2 are 26 | 2 are 28 | 2 are 30 | 2 are 32 | 2 are 34 | 2 are 36 | 2 are 38 |
| 3 — 39 | 3 — 42 | 3 — 45 | 3 — 48 | 3 — 51 | 3 — 54 | 3 — 57 |
| 4 — 52 | 4 — 56 | 4 — 60 | 4 — 64 | 4 — 68 | 4 — 72 | 4 — 76 |
| 5 — 65 | 5 — 70 | 5 — 75 | 5 — 80 | 5 — 85 | 5 — 90 | 5 — 95 |
| 6 — 78 | 6 — 84 | 6 — 90 | 6 — 96 | 6 — 102 | 6 — 108 | 6 — 114 |
| 7 — 91 | 7 — 98 | 7 — 105 | 7 — 112 | 7 — 119 | 7 — 126 | 7 — 133 |
| 8 — 104 | 8 — 112 | 8 — 120 | 8 — 128 | 8 — 136 | 8 — 144 | 8 — 152 |
| 9 — 117 | 9 — 126 | 9 — 135 | 9 — 144 | 9 — 153 | 9 — 162 | 9 — 171 |

43. The multiplication of one quantity by another is expressed by \times ; thus $7 \times 9 = 63$, means that 7 multiplied by 9 is equal to 63.

44. Quantities connected by the sign of multiplication are multiplied by any number, if we multiply any one of the factors by that number; thus $(9 \times 10 \times 2) \times 27 = 9 \times 10 \times 54$, or $9 \times 270 \times 2$; that is, if we multiply the factor 2 or the factor 10 by 27, we, in effect, multiply the whole number $(9 \times 10 \times 2)$ by 27.

45. When a quantity within brackets, consisting of several terms connected by the signs + and —, is to be multiplied by any number, each of its parts or terms must be multiplied. This arises from the fact that we consider the several terms within the bracket as constituting but one quantity, and to multiply the whole, we must multiply each of its parts. Thus $(7+8-3) \times 3 = 7 \times 3 + 8 \times 3 - 3 \times 3$; and $(8+7-5) \times (13-2)$ means that each of the terms within the former bracket is to be multiplied by each of the terms within the latter, or by their difference.

46. Let it be required to multiply 768 by 9.

Now $768 \times 9 = (700+60+8) \times 9 = 700 \times 9 + 60 \times 9 + 8 \times 9$ (Art. 44). Hence so far as the result is concerned, it matters not whether we commence multiplying at the lowest or at the highest denomination. $700 \times 9 + 60 \times 9 + 8 \times 9$ being evidently equal to $8 \times 9 + 60 \times 9 + 700 \times 9$.

Commencing the multiplication at the left-hand side, or highest denomination, the work is as follows:

| OPERATION. | | EXPLANATION.—7 hundreds multiplied by 9, or | |
|------------|-------------|---|---|
| 768 | which may | 768 | taken 9 times, are 63 hundreds; 6 tens multiplied |
| 9 | be thus ab- | 9 | by 9, are 54 tens; and 8 units multiplied by 9, are |
| — | breveiated. | — | 72 units. 63 hundreds, 54 tens, and 72 units, added |
| 6300 | | 63 | together, make 6912. The second operation shows |
| 540 | | 54 | the only abbreviation possible when we commence |
| 72 | | 72 | at the highest denomination. |
| — | | — | |
| 6912 | | 6912 | |

Let us now take the same question and commence at the right hand or lowest denomination.

| OPERATION. | | EXPLANATION.—No. II. | |
|------------|-------------|----------------------|--|
| I. 768 | which may | II. 768 | differs from No. I. only in |
| 9 | be thus ab- | 9 | having the unnecessary 0's |
| — | breveiated. | — | omitted. In No. III. the |
| 72 | | 72 | principle of <i>carrying</i> is taken |
| 540 | | 54 | advantage of, thus—8 units, |
| 6300 | | 63 | multiplied by 9, are 72 units, |
| — | | — | equal to 2 units and 7 tens, to |
| 6912 | | 6912 | carry—6 tens, multiplied by |
| | | | 9, are 54 tens, and 7 tens, |
| | | | make 61 tens, equal to 1 ten, |
| | | | and 6 hundreds to carry; 7 hundreds, multiplied by 9, are 63 hundreds, and |
| | | | 6 hundreds, make 69 hundreds, equal to 6 thousands and 9 hundreds. |

Hence, in order that we may be enabled to take advantage of the principle of *CARRYING*, we commence the multiplication at the right-hand or lowest denomination.

47. From the last article (46), for multiplying by any integral multiplier, not exceeding 12, (or 20 if the extended Multiplication Table be used) we deduce the following:—

RULE.

Multiply every order of units in the multiplicand in succes-

tion, beginning with the lowest, by the multiplier, and divide each product, so formed, by the number of that denomination which makes one unit of the next higher; write down each remainder under units of its own order, and carry the quotient to the next product.

EXAMPLE 1.—Multiply \$7896.43 by 11.

OPERATION. EXPLANATION.—3 hundredths of dollars, or cents, multiplied by 11, make 33 hundredths, equal to 3 hundredths, to set down, and 3 tenths to carry; 4 tenths of dollars, or tens of cents, multiplied by 11, make 44 tenths of dollars, and 3 tenths we carried, make 47 tenths, equal to 7 tenths and 4 units to carry; 6 units, multiplied by 11, make 66 units, and 4 units we carried, make 70 units, equal to 0 units to set down and 7 tens to carry; 9 tens, multiplied by 11, make 99 tens, and 7 tens, make 106 tens, equal to 6 tens and 10 hundreds; 8 hundreds, multiplied by 11, make 88 hundreds, and 10, make 98 hundreds, equal to 8 hundreds and 9 thousands; 7 thousands, multiplied by 11 make 77 thousands, and 9, make 86 thousands, equal to 6 thousands and 8 tens of thousands.

EXAMPLE 2.—Multiply 3 cwt. 2 qrs. 11 lbs. 7 oz. 6 drs. by 7.

OPERATION. EXPLANATION.—7 times 6 drams are 42 drams, equal to 10 drams to set down and 2 oz. to carry; 7 times 7 oz. are 49 oz., and 2 oz., make 51 oz., equal to 5 oz. to set down and 3 lbs. to carry; 7 times 11 lbs. are 77 lbs., and 3 lbs., make 80 lbs., equal to 5 lbs. to set down and 3 qrs. to carry; 7 times 2 qrs. are 14 qrs., and 3 qrs., make 17 qrs., equal to 1 qr. to set down and 4 cwt. to carry; 7 times 3 cwt. are 21 cwt., and 4 cwt., make 25 cwt.

EXERCISE.

| | | | | |
|----------|-----------|----------|----------|----------|
| | (3) | (4) | (5) | (6) |
| Multiply | 48960 | 75460 | 678000 | 57800 |
| By | 5 | 9 | 8 | 6 |
| | <hr/> | <hr/> | <hr/> | <hr/> |
| | 244800 | | | |
| | (7) | (8) | (9) | (10) |
| Multiply | 5.2736 | 8.7563 | 0.21375 | 0.0067 |
| By | 2 | 4 | 6 | 8 |
| | <hr/> | <hr/> | <hr/> | <hr/> |
| | 10.5472 | | | |
| | (11) | (12) | (13) | (14) |
| Multiply | \$767.62 | \$672.56 | \$789.76 | \$573.46 |
| By | 2 | 2 | 6 | 5 |
| | <hr/> | <hr/> | <hr/> | <hr/> |
| | \$1535.24 | | | |
| | (15) | (16) | (17) | (18) |
| Multiply | 8663.42 | 738579 | 4716375 | 8429763 |
| By | 11 | 12 | 811 | 12 |
| | <hr/> | <hr/> | <hr/> | <hr/> |

19. Multiply £32 8s. 6½d. by 5. *Ans.* £162 2s. 8½.
 20. Multiply £43 11s 4¾d. by 8. *Ans.* 348 11s. 2d.
 21. Multiply £125 13s. 0¼d. by 12. *Ans.* £1507 16s. 3d.
 22. Multiply 10 cwt. 3 qrs. 5 lbs. by 3. *Ans.* 32 cwt. 1 qr. 15 lbs.
 23. Multiply 7 yds. 3 qrs. 1 na. by 7. *Ans.* 54 yds. 2 qrs. 3 na.
 24. Multiply 11 oz. 10 dwt. 19 grs. by 12.
Ans. 11 lbs. 6 oz. 9 dwt. 12 gr.

48. When the multiplier is a composite number, and can be resolved into two or more factors, neither of which is greater than 12, we deduce from (41) the following:—

RULE.

Multiply by each of the factors in succession and the last product will be the entire product sought.

EXAMPLE 1.—Multiply 3 hrs. 7 min. 14 sec. by 64.

| OPERATION. | | | |
|------------|------|------|---------|
| hrs. | min. | sec. | ×64=8×8 |
| 3 | 7 | 14 | |
| | | 8 | |
| 1 | 0 | 57 | 52 |
| | | 8 | 8 |

EXPLANATION.—Multiplying 3 hrs. 7 min. 14 sec. by 8, we obtain 1 day 0 hrs. 57 min. 52 sec., which we again multiply by 8, and obtain 8 days 7 hrs. 42 min. 56 sec., which is the product of 3 hrs. 7 min. 14 sec., by 8 times 8 or 64.

8 7 42 56 *Ans.*

EXAMPLE 2.—Multiply 796'437 by 132.

| OPERATION. | | |
|------------|------|--------|
| 796'437 | ×132 | =11×12 |
| 11 | | |
| 8760'807 | | |
| 12 | | |

EXPLANATION.—We first multiply the given number by 11, or, in other words, take it 11 times, and then take this result 12 times, which is evidently equivalent to taking the given number 12 times 11 or 132 times.

105129'634=12 times 11 times multiplicand.

EXAMPLE 3.—Multiply 16 cwt. 3 qrs. 11 lb. by 270.

| OPERATION. | | | |
|------------|------|-----|--------|
| cwt. | qrs. | lb. | 11×270 |
| | | 3 | |
| 50 | 2 | 8 | 9 |
| | | 9 | |
| 455 | 0 | 22 | 10 |
| | | 10 | |
| 4552 | 0 | 20 | |

EXPLANATION.—270=10 times 27 or 10×3×9. If, therefore, we take the given multiplicand three times, and then this product 9 times, and then this second product 10 times, it is evident we shall have, in effect, taken the given multiplicand 3×9×10 or 270 times.

EXERCISE.

25. Multiply \$169'78 by 36. *Ans.* \$6112'08.
 26. Multiply 796342'3 by 121. *Ans.* 96357418'3.
 27. Multiply \$33460 by 144. *Ans.* \$4818240.
 28. Multiply 735 by 648. *Ans.* 476280.
 29. Multiply £3 7s. 6d. by 18. *Ans.* £60 15s. 0d.

30. Multiply £5 14s. 6½d. by 22. *Ans.* £125 19s. 11d.
 31. Multiply £3 4s. 7d. by 810. *Ans.* £2615 12s. 6d.
 32. Multiply 11 cwt. 3 qrs. 14 lb. 7 oz. by 54.
 Ans. 642 cwt. 1 qr. 4 lbs. 10 oz.
 33. Multiply 26 bush. 3 pks. 1 gal. 1 qt. 1 pt. by 49.
 Ans. 1319 bush. 0 pks. 1 gal. 1 qt. 1 pt.
 34. Multiply 2 yds. 2 qrs. 2 na. 2 in. by 63.
 Ans. 168 yds. 3 qrs. 2 na. 0 in.
 35. Multiply 5 days 17 hrs. 33 min. 11 sec. by 288.
 Ans. 1650 days, 15 hrs. 16 min. 48 sec.

49. When the multiplicand is a denominate number and the multiplier is greater than 12, but not a composite number, we proceed according to the following:—

RULE.

Take the nearest composite number to the given multiplier, multiply successively by its factors and add to or subtract from the product so many times the multiplicand as the assumed composite number is less or greater than the given multiplier.

EXAMPLE 36.—Multiply £62 12s. 6d. by 76.

OPERATION.

| | | |
|-------|----|----|
| £ | s. | d. |
| 62 | 12 | 6 |
| | | 8 |
| <hr/> | | |
| 501 | 0 | 0 |
| | | 9 |

4509 0 0 = 72 times multiplicand.

250 10 0 = 4 times multiplicand.

£4759 10 0 = 76 times multiplicand.

Instead of multiplying as above, we might have multiplied by 7 and 10 and increased the result by 6 times the multiplicand, or we might have multiplied by 7 and 11, and decreased the result by once the multiplicand, &c.

EXAMPLE 37.—Multiply 17 lbs. 3 oz. 7 dr. 2 scr. 16 grs. by 789.

OPERATION,

| lb. | oz. | dr. | scr. | grs. | |
|-------|-----|-----|------|------|------------------------------|
| 17 | 3 | 7 | 2 | 16 | × 9 = 9 times multiplicand. |
| | | | | 10 | |
| <hr/> | | | | | |
| 173 | 3 | 7 | 1 | 0 | × 8 = 80 times multiplicand. |
| | | | | 10 | |
| <hr/> | | | | | |
| 1733 | 3 | 1 | 1 | 0 | |
| | | | | 7 | |
| <hr/> | | | | | |
| 12132 | 10 | 1 | 1 | 0 | = 700 times multiplicand. |
| 1386 | 7 | 2 | 2 | 0 | = 80 times multiplicand. |
| 155 | 11 | 7 | 1 | 4 | = 9 times multiplicand. |
| <hr/> | | | | | |
| 13073 | 5 | 3 | 1 | 4 | = 789 times multiplicand. |

EXPLANATION.—We divide the given multiplier into $700+80+9$, and obtain the 3 partial products, which we add together, for the entire product.

EXAMPLE 38.—Multiply 3 wks. 6 days 17 hrs. 21 min. 12 sec. by 4736.

OPERATION.

| wks. ds. h. min. sec. | wks. ds. h. min. sec. | |
|-----------------------|-----------------------|-------------------------------|
| 3 6 17 21 12 | 23 5 8 7 12 | = 6 times multiplicand. |
| <u>10</u> | | |
| 39 4 5 32 | 118 5 16 36 | 0 = 30 times multiplicand. |
| <u>10</u> | | |
| 396 0 7 20 | 2772 2 3 20 | 0 = 700 times multiplicand. |
| <u>10</u> | | |
| 3900 3 1 20 | 15841 5 5 20 | 0 = 4000 times multiplicand. |
| | <u>18756 4 9 23</u> | 12 = 4736 times multiplicand. |

EXAMPLE 39.—Multiply £47 16s 2d. by 5783.

$$5783 = 5 \times 1000 + 7 \times 100 + 8 \times 10 + 3.$$

OPERATION.

| £ s. d. | £ s. d. | |
|-----------|---------------------|---|
| 47 16 2 | 143 8 6 | = product by units of the multiplicand. |
| <u>10</u> | | |
| 478 1 8 | 3824 13 4 | = product by tens of the multiplicand. |
| <u>10</u> | | |
| 4780 16 8 | 33465 16 8 | = product by hundreds of the multiplicand. |
| <u>10</u> | | |
| 47808 6 8 | 239041 13 4 | = product by thousands of the multiplicand. |
| | <u>276475 11 10</u> | = product by entire multiplier. |

EXERCISE.

40. Multiply £12 2s. 4d. by 83. *Ans.* £1005 13s 8d.
 41. Multiply £963 0s. 0½d. by 999. *Ans.* 962040 2s. 5½d.
 42. Multiply £3 6s. 5½d. by 3178. *Ans.* £10556 18s. 4½d.
 43. Multiply 16 bush. 3 pks. 1 gal. by 678. *Ans.* 11441 bush. 1 pk. 0 gal.
 44. Multiply 23 m. 6 fur. 33 rds. 4 yds. by 247. *Ans.* 5892 m. 2 fur. 10 rds. 3½ yds.
 45. Multiply 3S. 16° 30' 45" by 721. *Ans.* 2559S. 25° 30' 45"

50. It may be proper here to caution the pupil against the absurd attempt to multiply one denominate number by another. Multiplication is merely a particular kind of addition, and when we are required to multiply a quantity by any number, we are simply required to repeat it as many times as there are units in the multiplier. It is evident, then, that to talk of multiplying £19 19s. 11½d., by £19 19s. 11½d., or, in other words, of adding or repeating £19 19s. 11½d. £19 19s. 11½d. times is simply ridiculous. Nevertheless, great pains have been taken to show that 2s. 6d. may be multiplied by 2s. 6d., and that the product will be either 3½d. or 6s. 3d!! Undoubtedly 2s. 6d. can be taken 2½ times, and the result will be 6s. 3d.; or it can be taken one-eighth,

of a time, and the result will be 3½d.; but this is a very different thing from taking it 2s. 6d. times. In fact it is quite as nonsensical to talk of taking 2s. 6d. 2s. 6d. times as it would be to talk of taking 6 lbs. of beef 6 lbs. of beef times; or, 7 bars of music 7 bars of music times, &c. Duodecimal multiplication, which is sometimes adduced, as a proof that one denominate number can be multiplied by another, affords no support whatever to the theory, as will be fully shown hereafter. (See Sect. III.)

51. Let it be required to multiply 729 by 478.

OPERATION; EXPLANATION.—From the preceding examples it is evident that when units are multiplied into any order whatever, the product will always be of that order. Here, then, we first multiply by the 8 units, as in (47). Next we multiply by the 7 tens, thus:—9 units, multiplied by 7 tens, give 63 tens, equal to 3 tens, which we set down in the column of tens, and 6 hundreds which we carry; 2 tens, multiplied by 7 tens, give 14 hundreds, and 6 hundreds which we carried, make 20 hundreds, equal to 0 hundreds to set down and 2 thousands to carry, &c. Next we multiply by the 4 hundreds as follows:—9 units, multiplied by 4 hundreds, give 36 hundreds, equal to 6 hundreds to set down in the hundreds column, and 3 thousands to carry, &c. Lastly, we add the several partial products together.

Hence, when the multiplicand is an abstract number, the multiplier being greater than 12 and not a composite number, we have the following:—

RULE.

Multiply the multiplicand by each figure of the multiplier separately, beginning with the lowest, and write the partial products in separate lines, placing the first figure of each line directly under the figure by which you multiply, and, lastly, add the several partial products together.

EXAMPLE 46.—Multiply 7423 by 6709.

OPERATION. EXPLANATION.—Here, as there are no tens in the multiplier, we may either proceed directly to the hundreds after multiplying by the units, or we may set down a 0 under the tens, and then write the product by the hundreds in the same line, always remembering to place the first digit of the partial product under the figure by which we are multiplying in order that all the digits of the same order may come in the same vertical column.

EXERCISE.

| | (47) | (48) | (49) | (50) | (51) |
|----------|------|------|------|------|------|
| Multiply | 325 | 765 | 732 | 997 | 667 |
| By | 95 | 765 | 456 | 345 | 347 |

52. Multiply 7071 by 556.

Ans. 3931476.

53. Multiply 15607 by 3094.

Ans. 48288058.

54. Multiply 39948123 by 6007.

Ans. 239968374861.

55. Multiply 2778588 by 9867.

Ans. 27416327796.

52. Let it be required to multiply 63·5 by ·97.

OPERATION. EXPLANATION.—Since (51) any order, multiplied by units, will give that order—tenths, multiplied by units, will give tenths. Hence it is obvious that tenths, multiplied by tenths, will give the next lower order, or hundredths; and also that tenths, multiplied by hundredths, will give the next lower order again, or thousandths. In the above example, therefore, we proceed thus:—5 tenths, multiplied by 7 hundredths, give 35 thousandths, equal to 5 thousandths to set down and 3 hundredths to carry; 3 units, multiplied by 7 hundredths, give 21 hundredths, and 3 hundredths we carried, make 24 hundredths, equal to 4 hundredths to set down and 2 tenths to carry; 6 tens, multiplied by 7 hundredths, give 42 tenths, and 2 tenths we carried, make 44 tenths, equal to 4 tenths and 4 units. Again, 5 tenths, multiplied by 9 tenths, give 45 hundredths, equal to 6 hundredths to set down and 4 tenths to carry, &c.

53. Strictly speaking, all examples in multiplication of decimals should be worked according to the above method. An attentive consideration of the reasonings in (52) will, however, show that the lowest digit of the product of any two numbers containing decimals, must always be a number of places to the right of the decimal point, equal to the *sum* of the decimal places, in both multiplicand and multiplier.

Hence, when the multiplicand or multiplier, or both, contain decimals, we deduce the following—

RULE.

Multiply as though there were no decimals, and then remove the decimal point in the product as many places to the left as there are decimals in both the multiplicand and the multiplier.

EXAMPLE 56.—Multiply 5·63 by 0·00005.

OPERATION. EXPLANATION.—We multiply 563 by 5, and remove the decimal point *seven* places to the left, since there are *five* decimal places in the multiplier and *two* in the multiplicand, that is, we have taken a number a hundred times too great a hundred thousand times too often, and the product 2815 is therefore ten million times too great, and to make it what it should be, we divide it by ten millions; or, in other words, remove the decimal point seven places to the left.

EXAMPLE 57.—Multiply 2·073 by 5·12.

OPERATION. EXPLANATION.—We multiply as though both were whole numbers, and cut off *five* decimals, since there are *three* in the multiplicand and *two* in the multiplier.

EXERCISE.

| | (58) | (59) | (60) |
|----------|--------------------|----------|---------|
| Multiply | .003296 | 41.78 | 36.1234 |
| By | 5.782 | .0629 | 2.0006 |
| | Product .019057472 | 2.627962 | |

- | | |
|--------------------------------|------------------------|
| 61. Multiply 3.2517 by .023. | <i>Ans.</i> .0747891. |
| 62. Multiply 64.001 by 340. | <i>Ans.</i> 21760.34. |
| 63. Multiply 482000 by .37 | <i>Ans.</i> 178340. |
| 64. Multiply 3782.4 by .00917. | <i>Ans.</i> 34.684608. |
| 65. Multiply 87.96 by 220. | <i>Ans.</i> 19351.2. |

PROOF OF MULTIPLICATION.

54. *If the multiplier is not greater than 12, multiply the multiplicand by the multiplier, minus one, and add the multiplicand to the product. The sum should be the same as the product of the multiplicand of the whole multiplier.*

If the multiplier be greater than 12 and the multiplicand an abstract number :—

FIRST METHOD.—*Multiply the multiplier by the multiplicand and if the product thus obtained agree with the other the work may be considered correct.*

This method of proof depends upon the principle (40) that the product of two numbers is the same whichever is taken as multiplier.

SECOND METHOD.—*Divide the product by one of the factors, and if the quotient thus obtained is equal to the other factor, the work is correct.*

This is simply reversing the operation, i. e., breaking up the product into its factors.

THIRD METHOD.—*Divide the sum of the digits of the multiplicand by 9 and set down the remainder; divide also the sum of the digits of the multiplier by 9 and set down the remainder; multiply these two remainders together, divide the sum of the digits in their product by 9, and if the remainder thus obtained is equal to the remainder obtained by dividing the sum of the digits in the product of the multiplicand and the multiplier by 9, the work is generally correct: if these two last remainders are different, it must be wrong.*

EXAMPLE 1.—Let the quantities multiplied be 9426 and 3785.

Taking the nines from 9426, we get 3 as remainder.

And from 3785, we get 6.

$$\begin{array}{r}
 47130 \\
 75408 \quad 3 \times 5 = 15, \text{ from which 9 being taken, 6 are left.} \\
 \hline
 65982 \\
 28278 \\
 \hline
 \end{array}$$

Taking the nines from 36677410, 6 are left.

The remainders being equal, we are to presume the multiplication is correct. The same result, however, would have been obtained even if we had displaced digits, added or omitted cyphers, or fallen into errors which had counteracted each other; but, with ordinary care, none of these are likely to occur.

EXAMPLE 2.—Let the numbers be 76542 and 8436.

Taking the nines from 76542, the remainder is 6.
Taking them from 8436, it is 3.

$$\begin{array}{r} 459252 \\ 229626 \\ 306168 \\ \hline 612336 \end{array} \quad 6 \times 3 = 18, \text{ the remainder from which is 0.}$$

Taking the nines from 645703312 also, the remainder is 0.

The remainders being the same, the multiplication may be considered correct.

NOTE.—This proof applies, whatever may be the position of the decimal point in either of the given numbers.

EXAMPLE 3.—Let the numbers be 4.63 and 5.4.

From 4.63, the remainder is 4.
From 5.4, it is 0.

$$\begin{array}{r} 1852 \\ 2315 \\ \hline \end{array} \quad 4 \times 0 = 0, \text{ from which the remainder is 0.}$$

From 25.002 the remainder is 0.

55. The principle on which this process depends is, that if any number be divided by 9, and the sum of its digits also be divided by 9, the remainders are, in both cases, the same.

Thus, taking the number 7825, we have

$$\begin{aligned} 7825 &= \frac{7000 + 800 + 20 + 5}{9} = \frac{7000}{9} + \frac{800}{9} + \frac{20}{9} + \frac{5}{9} \\ &= 7 \times \frac{1000}{9} + 8 \times \frac{100}{9} + 2 \times \frac{10}{9} + \frac{5}{9} \\ &= 7 \times (111 + \frac{1}{9}) + 8 \times (11 + \frac{1}{9}) + 2 \times (1 + \frac{1}{9}) + \frac{5}{9} \\ &= 777 + 7 + 88 + \frac{8}{9} + 2 + \frac{2}{9} + \frac{5}{9} \\ &= 777 + 88 + 2 + \frac{7}{9} + \frac{8}{9} + \frac{2}{9} + \frac{5}{9} \\ &= 777 + 88 + 2 + \frac{7+8+2+5}{9} \end{aligned}$$

Hence the remainder arising from the division of 7825 by 9 is evidently the same as that arising from dividing $7+8+2+5$ or 22, which is the sum of its digits by 9.

56. Casting the nines from the factors, multiplying the resulting remainders, and casting the nines from the product, will leave the same remainder as if the nines were cast from the product of the factors—provided the multiplication has been correctly performed.

Thus, let the factors be 573 and 464.

Casting the nines from $5+7+3$ (which we have just seen is the same as casting the nines from 573), we obtain 6 as remainder. Casting the nines from $4+6+4$, we get 5 as remainder. Multiplying 6 and 5 we obtain 30 as product, which, when the nines are taken away, will give 3 as a remainder.

We can show that 3 will be the remainder, also, if we cast the nines from the product of the factors;—which is effected by setting down this product, and taking, in succession, quantities that are equal to it—as follows:

$$\begin{aligned} 573 \times 464 &= (\text{the product of the factors}), \\ &= (5 \times 100 + 7 \times 10 + 3) \times (4 \times 100 + 6 \times 10 + 4) \\ &= \left\{ 5 \times (99+1) + 7(9+1) + 3 \right\} \times \left\{ 4 \times (99+1) + 6 \times (9+1) + 4 \right\} \\ &= (5 \times 99 + 5 + 7 \times 9 + 7 + 3) \times (4 \times 99 + 4 + 6 \times 9 + 6 + 4) \end{aligned}$$

5×99 expresses a number of nines: it will continue to do so when multiplied by all the quantities within the second brackets, and is, therefore, to be cast out; and, for similar reason, 7×9 . Again 4×99 expresses a number of nines; it will continue to do so when multiplied by the quantities within the first brackets, and is therefore to be cast out; and for a similar reason, 6×9 . There will then be left only $(5+7+2) \times (4+6+4)$ —from which the nines are still to be cast out, the *remainders* to be multiplied together, and the nines to be cast from their product;—but we have done all this already, and obtained 3 as remainder.

CONTRACTIONS IN MULTIPLICATION.

57. To Multiply by 5:

Affix a 0 to the multiplicand and divide the result by 2.

$$\text{Reason } 5 = \frac{1}{2}.$$

II. Multiply by 15:

Affix a 0 to the multiplicand and to the result add half of itself.

$$\text{Reason } 15 = 10 + \frac{1}{2}.$$

III. To multiply by 25:

Affix two 0s to the multiplicand and divide the result by 4.

$$\text{Reason } 25 = \frac{1}{4}.$$

IV. To multiply by 125:

Affix three 0s to the multiplicand and divide the result by 8.

$$\text{Reason } 125 = \frac{1}{8}.$$

V. To multiply by 75:

Affix two 0s to the multiplicand and from the result take one-fourth of itself.

$$\text{Reason } 75 = 100 - \frac{1}{4}.$$

VI. To multiply by 175:

Affix two 0s—multiply the result by 7 and divide by 4.

$$\text{Reason } 175 = \frac{7}{4}.$$

VII. To multiply by 275:

Affix two 0s—multiply the result by 11 and divide by 4.

$$\text{Reason } 275 = \frac{11}{4}.$$

VIII. To multiply by 13, 14, 15, &c., or by 1 with either of the other digits *affixed* to it:

EXAMPLE. *Multiply by the units' figure of the multiplier, and write each figure of the partial product one place to the right of that from which it arises; finally, add the partial product to the multiplicand, and the result will be the answer required.*

Ans. 30225

REASON.—This is the same in effect as if we actually multiplied by the common method. We merely make the multiplicand serve for the second partial product.

IX. To multiply by 21, 31, 41, &c., or by 1 with either of the other significant figures prefixed to it:

EXAMPLE. *Multiply by the tens' figure of the multiplier, and write the first figure of the partial product in the tens' place; finally, add this partial product to the multiplicand, and the result will be the answer required.*

Ans. 7665

REASON.—The reason of this method of contraction is substantially the same as that of the preceding.

X. To multiply by 101, 102, 103, 104, &c., or by 10 with either of the other digits affixed to it:

Multiply by the units' figure of the multiplier and write the partial product, thus obtained, two places to the right of the multiplicand—finally, add the partial product to the multiplicand.

REASON.—Substantially the same as No. 8.

XI. To multiply by any number of nines:

Remove the decimal point of the multiplicand so many places to the right (by affixing 0's if necessary) as there are nines in the multiplier; and subtract the multiplicand from the result.

EXAMPLE 1.—Multiply 7347 by 999.

$$7347 \times 999 = 7347000 - 7347 = 7339653.$$

We, in such a case, merely multiply by the next higher convenient composite number, and subtract the multiplicand as many times as we have taken it too often; thus, in the example just given—

$$7347 \times 999 = 7347 \times (1000 - 1) = 7347000 - 7347 = 7339653.$$

EXAMPLE 2.—Multiply 678943 by 999999.

$$\begin{array}{r} 678943 \times 1000000 = 678943000000 \\ 678943 \times 1 = 678943 \\ \hline \end{array}$$

$$678943 \times 999999 = 678942321057$$

EXAMPLE 3.—Multiply 789645 by 99993

$$\begin{array}{r} 789645 \times 100000 = 78964500 \\ 789645 \times 7 = 5527515 \\ \hline \end{array}$$

$$789645 \times 99993 = 78958972435$$

XII. When it is not necessary to have as many decimal places in the product, as are in both multiplicand and multiplier—

Reverse the multiplier, putting its units' place under the place of that denomination in the multiplicand, which is the lowest of the required product.

Multiply by each digit of the multiplier, beginning with the denomination over it in the multiplicand; but adding what would have been obtained, on multiplying the preceding digit of the multiplicand—unity, if the number obtained would be between 5 and 15; 2, if between 15 and 25; 3, if between 25 and 35, &c.

Let the lowest denominations of the products, arising from the different digits of the multiplicand, stand in the same vertical column.

Add up all the products for the total product; from which cut off the required number of decimal places.

EXAMPLE 1.—Multiply 5.6784 by 9.7324, so as to have four decimals in the product.

| Short method. | Ordinary Method. |
|---------------|------------------|
| 56784 | 5 6784 |
| 42379 | 9 7324 |
| 511056 | 22 7136 |
| 39749 | 113 568 |
| 1703 | 1703 52 |
| 113 | 39748 8 |
| 22 | 511056 |
| 55 2643 | 55 2644 6016 |

9 in the multiplier expresses units; it is therefore put under the *fourth* decimal place of the multiplicand—that being the place of the lowest decimal required in the product.

In multiplying by each succeeding digit of the multiplier we neglect an additional digit of the multiplicand; because, as the multiplier decreases, the number multiplied must increase—to keep the lowest denomination of the different products, the same as the lowest denomination required in the *total* product. In the example given, 7 (the second digit of the multiplier) multiplied by 8 (the second digit of the multiplicand) will evidently produce the same denomination as 9 (one denomination higher than the 7), multiplied by 4 (one denomination lower than the 8). Were we to multiply the lowest denomination of the multiplicand by 7, we should get (53) a result in the *fifth* place to the right of the decimal point; which is a denomination supposed to be, in the present instance, too inconsiderable for notice—since we are to have only *four* decimals in the product. But we add unity for every ten that would arise, from the multiplication of an additional digit of the multiplicand; since every such *ten* constitutes *one* in the lowest denomination of the required product. When the multiplication of an additional digit of the multiplicand would give *more* than 5, and *less* than 15, it is nearer to the truth to suppose we have ten than either 0 or 20; and therefore it is more correct to add 1 than either 0 or 2. When it would give more than 15 and less than 25, it is nearer to the truth to suppose we have 20, than either 10 or 30; and therefore it is more correct to add 2 than 1 or 3; &c. We may consider 5 *either* as 0 or 10; 15 *either* as 10 or 20; &c.

On inspecting the results obtained by the abridged, and ordinary methods, the difference is perceived to be inconsiderable. When greater accuracy is desired, we should proceed as if we intended to have more decimals in the product, and afterwards reject those that are unnecessary.

EXAMPLE 2.—Multiply 8·76532 by 0·5764, so as to have three decimal places.

$$\begin{array}{r}
 876532 \\
 4675 \\
 \hline
 4383 \\
 613 \\
 52 \\
 3 \\
 \hline
 5051
 \end{array}$$

There are no units in the multiplier; but, as the rule directs, we put its units' *place* under the third decimal place of the multiplicand. In multiplying by 4, since there is no digit over it in the multiplicand, we merely set down what would have resulted from the multiplying the preceding denomination of the multiplicand.

EXAMPLE 3.—Multiply 0·23257 by 0·243, so as to have four decimal places.

$$\begin{array}{r}
 23257 \\
 342 \\
 \hline
 465 \\
 98 \\
 7 \\
 \hline
 00565
 \end{array}$$

We are obliged to place a cipher in the product to make up the required number of decimals.

EXERCISE.

66. The canals in Canada amount to 216 miles in length, and their average cost was \$83469 per mile. What was the total cost of the canals of Canada?

67. The Great Western Railroad is 229 miles in length, and its cost was about \$61135·37 per mile. What was the total cost of this road?

68. The Austrian empire contains 255226 square miles, and the population averages 143 per square mile. What is the entire population of the Austrian empire?

69. France contains 203736 square miles, and the population averages 176 per square mile. What is the entire population of France?

70. Great Britain contains 116700 square miles, and the population averages 235 per square mile. What is the entire population of Great Britain?

71. The total number of Common Schools in operation in Canada West, during the year 1857, was 3721; allowing an average of 73 pupils to each, how many children were in attendance at the Common Schools?

72. 32000 seeds have been counted in a single poppy; how many would be found in 297 of these?

73. 9344000 eggs have been found in a single cod fish; how many would there be in 35 such?

74. Multiply 123 lbs. 4 oz. 7 drs. 2 scr. 17 gr. by 749.
 75. Multiply 1698732 by 999998.
 76. Multiply 123 bush. 1 pk. 1 gal. 1 qt. 1 pt. by 640.
 77. What will be the cost of a chest of tea containing 89 lbs. at 73 cents per lb.?
 78. How much cloth will it take to make the clothes for a regiment of soldiers containing 1143 men, if each suit requires 7 yds. 3 qrs. 2 na. 1 in.?
 79. Multiply 1634·5789 by 635000.
 80. A person dying bequeathed the whole of his property to his three sons. To the youngest he gave \$968·49; to the second, 3·4 times as much as the youngest; and to the eldest 3·7 times as much as to the second. Required the value of his property.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers after the questions refer to the articles of the section.*

1. What is multiplication? (33)
2. What is the multiplicand? (34)
3. What is the multiplier? (35)
4. What is the product? (36)
5. Why are the multiplier and multiplicand called the factors of the product? (37)
6. What is a prime number? (38)
7. What is a composite number? (39)
8. If the multiplier be greater than unity, how will the product compare with the multiplicand? (39)
9. If the multiplier be equal to unity, how will the product compare with the multiplicand? (39)
10. If the multiplier be less than unity, how will the product compare with the multiplicand? (39)
11. Show that either of the factors may be used as multiplier without altering the value of the product. (40)
12. Show that when the multiplier is a composite number we may obtain the entire product by multiplying by each of the factors in succession. (41)
13. By whom was the multiplication table calculated? (42)
14. How was it calculated? (42)
15. What is the sign of multiplication? (43)
16. How do we multiply a quantity consisting of several factors connected by the sign of multiplication? (44)
17. How do we multiply a quantity consisting of several terms, connected by the signs + and — inclosed within a bracket? (45)
18. What is meant by $(7+3-2+5) \times (9+3-7)$? (45)
19. Why do we begin multiplying a number at the right-hand side? (46)
20. What is the rule for multiplication when the multiplier is not greater than 12? (47)
21. What is the rule when the multiplier is a composite number? (48)
22. What is the rule when the multiplicand is a denominate number and the multiplier greater than 12, but not a composite number? (49)
23. Show the absurdity of attempting to multiply one denominate number by another. (50)
24. When the multiplicand is an abstract number and the multiplier greater than 12, but not a composite number, what is the rule? (51)
25. When the multiplicand or multiplier, or both, contain decimals, what is the rule? (53)

26. Give the reason of this rule. (52 and 53)
 27. How do we prove multiplication when the multiplier is less than 12? (54)
 28. How do we prove multiplication when the multiplicand is an abstract number and the multiplier is greater than 12? (54)
 29. Upon what does the proof by casting out the nines depend? (55)
 30. Prove this principle. (55)
 31. Prove that casting the nines from the factors, multiplying the resulting remainders, and casting the nines from the product, will leave the same remainder as if the nines were cast from the product of the factors. (56)
 32. What short methods have we for multiplying by 5, 25 and 125? (57)
 33. What short methods of multiplying by 15 and 75? (57)
 34. How may we multiply by 175? How by 275? (57)
 35. How may we multiply by 13, 14, 15, &c? How by 101, 102, 103, &c? (57)
 36. How may we multiply by 21, 31, 41, &c? (57)
 37. How may we multiply by any number of nines? (57)
 38. How may we contract the work when we require only a limited number of decimals? (57)

DIVISION.

58. Division is the process of finding how many times one number is contained in another.

59. The number by which we divide is called the *divisor*.

60. The number to be divided is called the *dividend*.

61. The number obtained by division, that is, the number which shows *how many times* the divisor is contained in the dividend, is called the *quotient* (Lat. *quoties*, "how many times.")

62. If the divisor be less than the dividend, the quotient will be greater than unity.

If the divisor be equal to the dividend, the quotient will be equal to unity.

If the divisor be greater than the dividend, the quotient will be less than unity.

63. It is sometimes found that the dividend does not contain the divisor an *exact* number of times; in such cases the quantity left after the division is called the *remainder*.

The remainder, being a part of the dividend, is, of course, of the same denomination.

The remainder must be *less* than the divisor—otherwise the divisor would be contained *once more* in the dividend.

64. Division is merely a short method of performing a particular kind of subtraction (Art. 6, Sec. II.)—The dividend corresponds to the minuend, the divisor to the

subtrahend, and the remainder to the difference. The *quotient* has no corresponding quantity in subtraction—since it simply tells how many times the divisor can be subtracted from the dividend.

It will help us to understand how greatly division abbreviates subtraction, if we consider how long a process would be required to discover—by actually subtracting it—how often 7 is contained in 566495724, while, as we shall find, the same thing can be effected by *division* in less than a minute.

65. Since the quotient shows how many times the dividend contains the divisor, it follows that the divisor and quotient are the *factors* of the dividend. Hence, if the divisor and quotient be multiplied together, and the remainder, if any, added to the product, the result will be equal to the dividend.

66. We have three ways of expressing the division of one quantity by another:—

1st. By the sign \div , written between them; thus, $15 \div 3 = 5$.

2nd. By the sign : written between them; thus, $15 : 3 = 5$.

3rd. By writing the dividend above and the divisor below a horizontal line; thus, $\frac{15}{3} = 5$.

Two quantities written thus $\frac{17}{3}$ constitute what is called a fraction and the expression is read *sevenths*.

It is usual and proper to write the remainder obtained in division, in the form of a fraction; thus $17 \div 3$ gives 5 as a quotient and 2 as a remainder. Now the remainder, 2, is written above the line, and divisor, 3, below the line; the whole quotient being expressed thus $5\frac{2}{3}$ (read five and two-thirds). The meaning of which is, that 3 is contained in 17, 5 times and $\frac{2}{3}$ of a time.

67. When a quantity consisting of several terms connected by the sign of multiplication is to be divided, dividing any one of the factors will be the same as dividing the product; thus $5 \times 10 \times 25 \div 5 = \frac{5}{5} \times 10 \times 25$ for each is equal to 250.

68. When a quantity consisting of several terms, connected by the signs + and —, contained within brackets, is to be divided, it is necessary, on removing the brackets, to put the divisor under each of the terms of the quantity;

thus $(6+3-7+9) \div 3$, or $\frac{6+3-7+9}{3} = \frac{6}{3} + \frac{3}{3} - \frac{7}{3} + \frac{9}{3}$; for

we do not divide the whole unless we divide *all* its parts.

69. It will be seen from (68) that the horizontal line

which separates the dividend from the divisor assumes the place of a pair of brackets when the dividend consists of several terms; and, therefore, when the quantity to be divided is subtractive, it will sometimes be necessary to change the signs, as already directed (26); thus:

$$\frac{6}{2} + \frac{13-3}{2} = \frac{6+13-3}{2}; \text{ but } \frac{27}{3} - \frac{15-6+9}{3} = \frac{27-15+6-9}{3}$$

70. Let it be required to divide 798 by 3.

OPERATION. EXPLANATION.—Place the divisor a little to the left of the dividend and separate them by a short curve line. Also draw a straight line beneath the dividend.

$$\begin{array}{r} 3 \overline{)798} \\ \underline{266} \\ \text{Now } \frac{798}{3} = \frac{700+90+8}{3} = \frac{600+190+8}{3} = \frac{600+180+18}{3} = \frac{600}{3} + \frac{180}{3} + \frac{18}{3} \\ +60+6 = 266 \text{ (See 68 and 69).} \end{array}$$

Instead of going through this long operation, it is evident that we may proceed as follows: 3 units into 7 hundreds will go 3 (hundreds) times and leave a remainder 1, which, being of the order hundreds, is equal to 10 tens; 10 tens and 9 tens make 19 tens, and 3 into 19 goes 6 (tens) times and leaves a remainder 1, which, being of the order of tens, is equal to 10 units; 10 units and 8 units make 18 units, and 3 units into 18 units goes 6 (units) times.

2. Let it be required to divide 917 lb. 13 oz. 12 dr. by 4.

OPERATION. EXPLANATION.—Placing the dividend and divisor as before we proceed thus: 4 in 9, 2 (hundreds) times and 1 over; 1 hundred, equal to 10 tens, and 1 ten make 11 tens; 4 in 11, 2 (tens) times and 3 over; 3 tens, equal to 30 units, and 7 units make 37 units; 4 in 37, 9 times and 1 over, which is 1 lb. because the 917 are pounds (63); 1 lb., equal to 16 oz., and 13 oz. make 29 oz.; 4 in 29, 7 times and 1 over, which is 1 oz., since the 29 are oz.; 1 oz. is equal to 16 drams and 12 drams make 28 drams; 4 in 28, 7 times.

Observe that any order divided by units gives that order in the quotient.

3. Let it be required to divide 9789 by 26.

OPERATION. EXPLANATION.—Placing the dividend and divisor as before, we say 26 in 9 (thousands) no times; 26 in 97 (hundreds), 3 (hundreds) times. We place the 3 (hundreds) to the right of the dividend and, multiplying the divisor 26 by it, get 78 hundred, which we subtract from the 97 hundred, and obtain a remainder 19 hundreds. 19 hundreds are equal to 190 tens and 8 tens, make 198 tens; 26 in 198, 7 (tens) times. Multiplying the 26 by the 7 tens, we get 182 tens, which, subtracted from 198 tens, leaves a remainder of 16 tens. 16 tens are equal to 160 units and 9 units make 169 units; 26 in 169, goes 6 times, and leaves a remainder 13. This 13 should be divided by 26, but since 13 does not contain 26, the division cannot be effected, and we can only indicate it, which we do by placing the 26 under the 13, as is explained in (Art. 66).

The complete quotient is therefore 376 $\frac{13}{26}$ read 376 and thirteen-twenty-sixths or 376 and 13 divided by 26.

71. From the preceding illustration and examples we deduce, for the division of numbers, the following general

RULE.

Beginning with the highest order of units in the dividend, pass on to the lower orders until the fewest number of figures be found that will contain the divisor; divide these figures by it, for the first figure of the quotient; this figure will be of the same order as that of the lowest used in the partial dividend.

Multiply the divisor by the quotient figure so found, and subtract the product from the dividend, being careful to place units of the same order in the same vertical column. Reduce the remainder to units of the next lower order, and add in the units of that order found in the dividend: this will furnish a new dividend.

Proceed in a similar manner until units of every order shall have been divided.

EXAMPLE 1.—Divide 98765 by 7.

OPERATION. EXPLANATION.—Here we say 7 in 9, 1 and 2 over; in 28 4 and 0 over; in 7, 1 and 0 over; in 6, 0 times and 6 over; in 7)98765
14109 $\frac{7}{7}$ 65, 9 and 2 over. Beneath this 2 we write the divisor 7, to indicate its division. We may however, carry on the division by considering the 2 units reduced to tenths, &c., and the quotient becomes 14109 $\frac{2857}{7}$.

Thus 2 units, equal to 20 tenths, 7 in 20, 2 and 6 over; 6 tenths are equal to 60 hundredths, 7 in 60, 8 times and 4 over; 4 hundredths are equal to 40 thousandths, 7 in 40, 5 and 5 over; 5 thousandths are equal to 50 tenths of thousandths, &c.

EXAMPLE 2.—Divide 124789 by 12.

OPERATION. EXPLANATION.—Here again we may either stop at the 12)124789
10399 $\frac{17}{12}$ units and write the remainder 1 over the divisor 12, or we may reduce the 1 unit to tenths, &c., as in the second operation.

10399 $\frac{17}{12}$
or
12)124789

10399 $\frac{083}{12}$ +

EXAMPLE 3.—Divide £1986 14s. 7 $\frac{1}{2}$ d. by 9.

OPERATION. EXPLANATION.—9 in 19, 2 and 1 over; 9 in 18, 2 and 0 9)£1986 14 7 $\frac{1}{2}$ over; 9 in 6, 0 and 6 over; £8 are equal to 120s. and 14s. make 134s.; 9 in 134, 14 and 8 over; 8s. are equal to 220 14 11 $\frac{1}{2}$ to 96d. and 7d. make 103d.; 9 in 103, 11 times and 4 over; 4d. are equal to 16 farthings and 2 farthings make 18 farthings; 9 in 18, 2, i. e., one ninth of 18 farthings is 2 farthings, written thus $\frac{1}{2}$ d.

72. In example 3, we are, in reality, required to find one-ninth of the dividend. The obvious meaning is, not that 9 is contained in £1986 14s. 7 $\frac{1}{2}$ d. £220 14s. 11 $\frac{1}{2}$ d. times, which would be nonsense, but that £220 14s. 11 $\frac{1}{2}$ d. is the ninth part of £1986 14s. 7 $\frac{1}{2}$ d.: so also in all similar questions.

Notwithstanding this, all such examples are reducible to a species of subtraction. Thus, in the above example we for the

moment, consider the divisor 9 to be of the same denomination as the dividend, and ascertain how many times £9 will go into (i. e., can be subtracted from) £1986. We get, as a result, 220 times, and a remainder of £6. Then we argue, from the principles already established, that since £9 is contained in £1986 220 times, with a remainder of £6; £220 is contained in £1986 9 times, with a remainder of £6; that is, that the ninth part of £1986 is £220, with a remainder £6. Next reducing this £6 to shillings, and adding in the 14s., we obtain a total of 134s., and we find that 9s. is contained in 134s. 14 times, with a remainder of 8s., whence we conclude that 14s. is contained in 134s. 9 times, with a remainder of 8s., that is, that the ninth part of 134s. is 14s. with a remainder of 8s., or that the ninth part of £1986 14s. is £220 14s., with 8s. still undivided, &c.

EXAMPLE 4.—Divide 978964 by 3429.

| | |
|---------------------------------|------|
| OPERATION. | |
| 3429)978964(285 ¹⁶⁹⁹ | 3429 |
| 6858 | |
| 29316 | |
| 27132 | |
| 18844 | |
| 17145 | |
| 1699 | |

EXPLANATION.—3429 into 9789 (the smallest number of figures that will contain the divisor) goes 2 times, we therefore put 2 in the quotient. Multiplying 3429 by 2, we get 6858, which we subtract from 9789; and obtain as remainder 2931, which we reduce to the next lower order (tens) and add in the 6 tens, 3429 into 29316 goes 8 times. We therefore place 8 in the quotient. Multiplying 3429 by 8 we get 27432, which we subtract from 29316, and obtain 1884 as a remainder. Reducing to units and adding in the 4, or what amounts to the same thing, bringing down the 4 and writing it after the 1884 we get 18844; and

3429 into 18844 goes 5 times, with a remainder 1699, under which we write the divisor 3429.

73. When the dividend is an abstract number, it is evident that bringing down the next figure and writing it to the right of the remainder, is the same in effect as reducing the remainder to the next lower denomination and adding in the units of that order found in the dividend. Thus, in the last example, bringing down the 6 and writing it directly to the right of the first remainder, 2931, makes the next partial dividend 29316, which is the same as reducing the 2931 to the next lower order and adding to the result the 6 of that order found in the dividend.

EXAMPLE 5.—Divide 6421284 by 642.

| | |
|-------------------|--|
| OPERATION. | |
| 642)6421284(10002 | |
| 642 | |
| 1284 | |
| 1284 | |

EXPLANATION.—642 goes once into 642, and leaves no remainder. Bringing down the next digit of the dividend gives no digit in the quotient, in which, therefore, we put a cipher after the 1. The next digit of the dividend, in the same way, gives no digit in the quotient, in which, consequently, we put another cipher, and, for similar reasons, another in bringing down the next; but the next digit makes the quantity brought down 1284, which contains the divisor twice, and gives no remainder:—we put 2 in the quotient.

NOTE.—After the first quotient figure is obtained, for each figure of the dividend which is brought down, either a significant figure, or a cipher, must be put in the quotient.

74. When there is a remainder, we may continue the division, adding decimal places to the quotient, as follows—

EXAMPLE 6.—Divide 796347 by 847, and the result by 7234.

$$\begin{array}{r}
 \text{OPERATION.} \\
 847 \overline{)796347} (940 \text{ } 197166, \text{ \&c.} \\
 \underline{7623} \\
 3404 \\
 \underline{3388} \\
 1670 \\
 \underline{847} \\
 8230 \\
 \underline{7623} \\
 6070 \\
 \underline{5929} \\
 1410 \\
 \underline{847} \\
 5630 \\
 \underline{5082} \\
 5480 \\
 \underline{5082} \\
 398, \text{ \&c.} \\
 7234 \overline{)940197166} (0 \text{ } 129969, \text{ \&c.} \\
 \underline{7234} \\
 21679 \\
 \underline{14468} \\
 72117 \\
 \underline{65106} \\
 70111 \\
 \underline{65106} \\
 50055 \\
 \underline{43404} \\
 66525 \\
 \underline{65106} \\
 1420, \text{ \&c.}
 \end{array}$$

75. When the divisor is large, the pupil will find assistance in determining the quotient figure, by finding how many times the first figure of the divisor is contained in the first figure, or, if necessary, the first *two* figures of the dividend. This will give pretty nearly the right figure. Some allowance, must, however, be made for carrying from the product of the other figures of the divisor, to the product of the first into the quotient figure. After multiplying the divisor by the quotient figure, if the product is greater than the corresponding partial dividend, this shows the quotient was taken too great, and must be diminished. If the remainder, after subtraction, is greater than the divisor, the quotient was taken too small, and must be increased.

EXAMPLE 7.—Divide 279 cwt. 3 qrs. 14 lb. 9 oz. by 129.

| | | OPERATION. | | | | | | | | |
|------|-----------------|------------|----|---|---|---|----|----|---|-----|
| 129) | 279 | 3 | 14 | 9 | 2 | 0 | 16 | 15 | 9 | 23 |
| | 258 | | | | | | | | | 139 |
| | 21 | | | | | | | | | |
| | 4=qrs. in cwt. | | | | | | | | | |
| | 87 | | | | | | | | | |
| | 25=lbs. in qr. | | | | | | | | | |
| | 449 | | | | | | | | | |
| | 174 | | | | | | | | | |
| | 2189 | | | | | | | | | |
| | 129 | | | | | | | | | |
| | 899 | | | | | | | | | |
| | 774 | | | | | | | | | |
| | 125 | | | | | | | | | |
| | 16=oz. in lb. | | | | | | | | | |
| | 759 | | | | | | | | | |
| | 125 | | | | | | | | | |
| | 2009 | | | | | | | | | |
| | 129 | | | | | | | | | |
| | 719 | | | | | | | | | |
| | 645 | | | | | | | | | |
| | 74 | | | | | | | | | |
| | 16=drams in oz. | | | | | | | | | |
| | 444 | | | | | | | | | |
| | 74 | | | | | | | | | |
| | 1184 | | | | | | | | | |
| | 1161 | | | | | | | | | |
| | 23 remainder. | | | | | | | | | |

EXPLANATION.—129 in 279, i. e., the 129th part of 279 cwt., is 2 cwt., with a remainder of 21 cwt. This 21 cwt. we reduce to quarters by multiplying by 4 and adding in the 3 qrs. The 129th part of 87 qrs. is equal to 0 qr., and we therefore place a 0 in the quarters' place of the quotient. We next reduce qrs. to lbs. by multiplying by 25 and adding in the 14 lbs. of the dividend. We thus obtain 2189 lbs., of which the 129th part is 16 lb., with an undivided remainder of 125 lbs. Reducing 125 lbs. to oz., and adding in the 9 oz., we obtain 2009 oz., of which the 129th part is 15 oz., with an undivided remainder of 74 oz. Reducing the 74 oz. to drams, we obtain 1184 drams, of which the 129th part is 9 drams, with an undivided remainder of 23 drams, under which we place the divisor 129 to indicate its division. Thus we find the total quotient to be 2 cwt. 0 qr. 16 lb. 15 oz. 9 $\frac{23}{129}$ drs.

76. The general principles on which the operations in division depends are:—

1st. The quotient arising from the division of the whole dividend by the divisor, is equal to the sum of the quotients arising from the division of the several parts of the dividend by the divisor. (68)

2nd. The divisor and quotient are the factors of the dividend. (65)

3rd. The product of the divisor, by the entire quotient, is equal to the sum of the products of the divisor by the several parts of the quotient. (45)

We ask how many times the divisor is contained in a part of the dividend, and thus a part of the quotient is found; the product of the divisor by this part is taken from the dividend, showing how much of the latter remains undivided; then a part of the remaining dividend is taken and another part of the quotient is found, and the product of the divisor, by it, is taken away from what before remained; and thus the operation proceeds till the *whole* of the dividend is divided, or till the *remainder* is less than the *divisor*.

77. We begin at the left-hand side, because what remains of the higher denomination may still give a quotient in a lower; and the question is, *how often* the divisor will go into the dividend—its different denominations being taken in *any* convenient way. We cannot know how many of the higher we shall have to add to the lower denominations, unless we begin with the higher.

PROOF OF DIVISION.

78. FIRST METHOD.—*Multiply the quotient by the divisor, and to the product add the remainder, if any; the result should be equal to the dividend.* (65)

EXAMPLE 8.—Divide £5681 13s. 4d. by 700.

| £ | s. | d. | £ | s. | d. | PROOF. |
|-----|------|------|--------|----|----|--|
| 700 | 5681 | 13 4 | (8 2 4 | | | £ s. d. |
| | 5600 | | | | | 8 2 4 |
| | | | | | | 10 |
| | 81 | | | | | 10 |
| | 20 | | | | | 81 3 4 |
| | | | | | | 10 |
| | 1633 | | | | | |
| | 1400 | | | | | 811 13 4 |
| | | | | | | 7 |
| | 233 | | | | | |
| | 12 | | | | | |
| | | | | | | 5681 13 4 = £8 2s. 4d. × 700 = dividend. |
| | 2800 | | | | | |
| | 2800 | | | | | |

SECOND METHOD.—*Subtract the remainder, if any, from the dividend, divide the dividend, thus diminished, by the quotient; and if the result is equal to the given divisor, the work is right.*

This is merely doing the same work by a different method.

THIRD METHOD.—*Cast the nines out of the divisor and quotient and multiply the remainders together; add to their product the remainder, if any, after division, and cast the nines out of this sum; the remainder thus obtained should be equal to the remainder obtained by casting the nines out of the dividend.*

Since the divisor and quotient answer to the multiplier and multiplicand, and the dividend to the product, it is evident that the principle of casting out the 9s. will apply to the proof of division as well as to that of multiplication.

FOURTH METHOD.—*Add the remainder and the respective products*

of the divisor into each quotient figure together; and if the sum is equal to the dividend, the work is right.

This mode of proof depends upon the principle that the whole of a quantity is equal to the sum of all its parts.

EXAMPLE 9.—Divide 147856 by 97.

$$\begin{array}{r}
 97 \overline{)147856(1524} \\
 \underline{97} \\
 508 \\
 \underline{485} \\
 235 \\
 \underline{194} \\
 416 \\
 \underline{388} \\
 28 \\
 \underline{147856}
 \end{array}$$

NOTE.—The asterisks show the lines to be added.

EXERCISE.

- | | | | |
|------------------------------|-------------------------------|-----------------------------|--|
| (10) 12)876967 | (11) 7)891023 | (12) 9)763457 | (13) 8)65432·978 |
| <u>73080</u> ₁₂ | <u>127289</u> | <u>84828</u> ₉ | <u>8179·12225</u> |
| (14) \$ cts. 9)6789·60 | (15) \$ cts. 11)4298·76 | (16) £ s. d. 4)19 6 4 | (17) wks. ds. hrs. min. 9)69 4 19 30 |
| <u>\$754·40</u> | <u>\$390·79</u> ₁₁ | <u>4 16 7</u> | <u>7 5 4 50</u> |
18. Divide 798965 by 6423. *Ans.* 124 $\frac{2}{3}$ $\frac{1}{3}$.
19. Divide £176 14s. 6d. by 13. *Ans.* £14 14s. 6 $\frac{1}{2}$ d.
20. Divide 56789 by 741. *Ans.* 76 $\frac{1}{3}$ $\frac{1}{3}$.
21. Divide 6785158 by 7894. *Ans.* 859 $\frac{1}{3}$ $\frac{1}{3}$.
22. Divide £4728 16s. 2d. by 317. *Ans.* £14 18s. 4 $\frac{1}{3}$ $\frac{1}{3}$ d.
23. Divide \$97896·64 by 429. *Ans.* \$228·19 $\frac{1}{3}$ $\frac{1}{3}$.
24. Divide 970763 by 6. *Ans.* 161793·8333 $\frac{1}{3}$.
25. Divide 71234 by 9. *Ans.* 7914 $\frac{2}{3}$.
26. Divide 977076 by 47600. *Ans.* 20 $\frac{1}{4}$ $\frac{1}{10}$.
27. Divide 7289 lbs. 6 oz. 4 drs. 2 scr. 13 grs. by 498. *Ans.* 14 lbs. 7 oz. 5 dr. 0 oz. 12 $\frac{1}{3}$ $\frac{1}{3}$.
28. Divide £157 16s. 7d. by 487. *Ans.* 6s. 5 $\frac{1}{3}$ d. $\frac{1}{3}$.
29. Divide 7867674 by 9712. *Ans.* 810 $\frac{3}{4}$ $\frac{1}{4}$.
30. Divide 422 m. 3 fur. 38 rds. by 37. *Ans.* 11 m. 3 fur. 14 rds.

GENERAL PRINCIPLES.

79. If a given divisor is contained in a given dividend a certain number of times, the same divisor will be contained in *double* that dividend *twice* as many times; in *three* times that dividend *thrice* as many times, &c. Hence,

When the divisor remains the same, multiplying the dividend by any number has the effect of multiplying the quotient by the same number.

Thus $9 \div 3 = 3$; 9×2 or $18 \div 3 = 6 = 3 \times 2$. 9×5 or $45 \div 3 = 15 = 3 \times 5$, &c.

80. If a given divisor is contained in a given dividend a certain number of times, the same divisor will be contained in *half* that dividend *half* as many times; in *one-third* of that dividend *one-third* as many times, &c. Hence,

When the divisor remains the same, dividing the dividend by any number, has the effect of dividing the quotient by the same number.

Thus $48 \div 3 = 16$; $\frac{48}{2} \div 3$ or $24 \div 3 = 8 = \frac{16}{2}$; $\frac{48}{3} \div 3$ or $6 \div 3 = 2 = \frac{16}{3}$, &c.

81. If a given divisor is contained in a given dividend a certain number of times, *half* that divisor will be contained in the same dividend *twice* as many times, *one-third* of that divisor *thrice* as many times, &c. Hence,

When the dividend remains the same, dividing the divisor by any number has the effect of multiplying the quotient by that number.

Thus $48 \div 6 = 8$; $48 \div \frac{6}{2}$ or $48 \div 3 = 16 = 8 \times 2$; $48 \div \frac{6}{3}$ or $48 \div 2 = 24 = 8 \times 3$, &c.

82. If a given divisor is contained in a given dividend a certain number of times, *twice* that divisor will be contained in the same dividend only *half* as many times, three times that divisor only *one-third* as many times, &c. Hence,

When the dividend remains the same, multiplying the divisor by any number has the effect of dividing the quotient by the same number.

Thus $48 \div 2 = 24$; $48 \div$ twice 2 or $48 \div 4 = 12 =$ half of 24.
 $48 \div$ eight times 2 or $48 \div 16 = 3 =$ one-eighth of 24, &c.

83. If a given divisor is contained in a given dividend a certain number of times, *twice* that divisor is contained in *twice* that dividend the same number of times; *thrice* that divisor in *thrice* that dividend the same number of times, &c. Hence,

When the divisor and dividend are both multiplied by the same number, the quotient will remain unchanged.

Thus $12 \div 4 = 3$; 24 or twice $12 \div 8$ or twice $4 = 3$; 72 or thrice $24 \div 24$ or thrice $8 = 3$, &c.

84. If a given divisor is contained in a given dividend a certain number of times, *half* that divisor is contained in *half* that dividend the same number of times; *one-third* that divisor in *one-third* that dividend the same number of times, &c. Hence,

When the divisor and dividend are both divided by the same number, the quotient will remain unchanged.

Thus $48 \div 24 = 2$; 24 or half of $48 \div 12$ or half of $24 = 2$, &c.

TO DIVIDE BY A COMPOSITE NUMBER.

RULE.

85.—*Divide the dividend by one of the factors of the divisor; then the resulting quotient by another factor; and so on till all the factors are used. The last quotient will be the answer.*

Multiply each remainder by all the preceding divisors and add their products to the first remainder, if any, for the true remainder.

When the divisor is separated into only two factors, the rule for finding the true remainder may be thus expressed:—

Multiply the last remainder by the first divisor, and to their product add the first remainder, if any; the result will be the true remainder.

EXAMPLE 31.—Divide 718 lbs. by 72.

| | OPERATION. | |
|---------|----------------------|--------------------------------------|
| 3)718 | 1st remainder | = 1 lb. |
| 4)239—1 | 2nd remainder=3×3 | = 9 lb. |
| 6)69—3 | 3rd remainder=5×4×3= | 60 lb. |
| 9—5 | true remainder | 70 lb. <i>Ans.</i> 9 $\frac{7}{9}$. |

That dividing by the factors of a number will give the same quotient as dividing by the number itself, follows directly from (Art. 84).

In the last example, dividing by 3 distributes the 718 lbs. into 239 parcels of 3 lbs. each, and leaves a remainder of 1 lb.; dividing next by 4 distributes the 239 parcels into 59 still larger parcels, each containing 4 of the smaller or 3 lb. parcels, and leaves a remainder 3, which is not 3 lbs. but 3 parcels, each of 3 lb.; lastly, dividing the 59 by 6 distributes it into 9 large parcels of 72 lbs. each, and leaves a remainder 5, which is, of course, 5 of the 12 lb. parcels. Hence the reason of the rule for finding the true remainder.

EXERCISE.

- | | |
|--|--|
| 32. Divide 3766 by 25. | <i>Ans.</i> 150 $\frac{1}{5}$. |
| 33. Divide 26406 by 42. | <i>Ans.</i> 628 $\frac{2}{3}$. |
| 34. Divide 25431 by 96. | <i>Ans.</i> 264 $\frac{3}{8}$. |
| 35. Divide £24 17s. 6d. by 24. | <i>Ans.</i> £1 0s. 8 $\frac{1}{2}$ d. |
| 36. Divide £740 13s. 4d. by 49. | <i>Ans.</i> £15 2s. 3 $\frac{1}{2}$ d. $\frac{1}{4}$. |
| 37. Divide £547 12s. 4d. by 56. | <i>Ans.</i> £9 15s. 6 $\frac{1}{2}$ d. $\frac{1}{8}$. |
| 38. Divide 6789436 by 35. | <i>Ans.</i> 193983 $\frac{1}{5}$. |
| 39. Divide 753293 by 147 (=7×7×3) | <i>Ans.</i> 5124 $\frac{1}{7}$. |
| 40. Divide 1798 lbs. 6 oz. 11 dwt. 9 grs. by 81. | <i>Ans.</i> 22 lbs. 2 oz. 9 dwt. 0 $\frac{1}{3}$ grs |

86. When both the divisor and the dividend are denominate numbers—

RULE.

Reduce both the divisor and the dividend to the lowest denomination contained in either, and then proceed as in Art. 71.

EXAMPLE 41.—Divide £37 5s. 9½d. by 3s. 6½d.

| | |
|----------------|-------------------------------------|
| s. d. | £ s. d. |
| 3 6½ | 37 5 9½ |
| 12 | 20 |
| — | — |
| 42 | 745 |
| 4 | 12 |
| — | — |
| 170 farthings. | 8949 |
| | 4 |
| | — |
| | 170)35797 (210 ³⁷ times. |
| | 340 |
| | — |
| | 179 |
| | 170 |
| | — |
| | 97 |

87. In the above and all similar questions we are required to find what fraction the divisor is of the dividend; or, in other words, how often the divisor is contained in, or can be subtracted from, the dividend, and the quotient must necessarily be an *abstract* number.

EXAMPLE 42.—Divide 729 cwt. 3 qrs. 16 lb. by 3 qrs. 9 lb. 7 oz.

| | |
|---------------|---|
| qrs. lbs. oz. | cwt. qrs. lbs. |
| 3 9 7) | 729 3 16 |
| 25 | 4 |
| — | — |
| 84 | 2919 |
| 16 | 25 |
| — | — |
| 511 | 14611 |
| 84 | 5838 |
| — | — |
| 1351 oz. | 72991 |
| | 16 |
| | — |
| | 437946 |
| | 72991 |
| | — |
| | 1351)1167856 oz. (864 ⁵⁹² times. |
| | 10808 |
| | — |
| | 8705 |
| | 8106 |
| | — |
| | 5996 |
| | 5404 |
| | — |
| | 592 |

EXERCISE.

43. Divide £8968 13s. 7½d. by £491 12s. 0½d. *Ans.* 18½½½½.
 44. Divide 1027 m. 1 fur. 6 rds. by 17 m. 5 fur. 27 rds. *Ans.* 58.
 45. Divide £171 1s. 10½d. by £57 0s. 7½d. *Ans.* 3.
 46. Divide 9 lb. 9 oz. 3 dwt. 12 grs. by 5 dwts. 9 grs. *Ans.* 436.
 47. Divide 2366 acres 3 roods 36 rds. by 91 acres 6 rds. *Ans.* 26.

88. When the dividend alone contains decimal places, the preceding rules are sufficient; but when the divisor contains decimals, it becomes necessary to prepare the quantities for division according to the following—

RULE.

Remove the decimal point as many places to the right, in both the dividend and the divisor, as there are decimals in the divisor, and then proceed as in Art. 71.

This is simply multiplying both dividend and divisor by the same number, and therefore (Art. 83) does not affect the quotient. Thus removing the decimal point one place to the right, in both dividend and divisor, is equivalent to multiplying each by 10; two places, the same as multiplying each by 100; three places, by 1000, &c.

EXAMPLE 48.—Divide 87·6 by ·0009.

Multiplying each by 10000, or, in other words, removing the decimal point four places to the right, in each, (since there are *four* decimals in the divisor), gives us 876000 ÷ 9, and this (Art. 83) must give the same quotient as $87\cdot6 \div \cdot0009 = 876000 \div 9 = 97333\cdot33$, &c.

EXAMPLE 49.—Divide ·06 by 8·934.

$$\cdot06 \div 8\cdot934 = 60 \div 8934.$$

$$8934 \overline{)60'000(0'0067, \&c.}$$

53'604

6'3960

6'2538

1422

Removing the decimal point *three* places to the right, in each, we get 60 ÷ 8934, and we then proceed thus: 8934 into 60 (units), 0 (units) times; set down 0 with the decimal point after it; 8934 into 600 (tenths), 0 times; into 6000 (hundredths), 0 times; into 60000 (thousandths), 6 (thousandths) times, &c.

EXAMPLE 50.—Prepare 93·004 ÷ ·0000069 for division.

$$\text{Ans. } 93\cdot004 \div \cdot0000069 = 930040000 \div 69.$$

EXERCISE.

51. $43 \div \cdot0006947 = 430000000 \div 6947.$
 52. $9378\cdot92 \div 9\cdot7891 = 93789200 \div 97891.$
 53. $4\cdot96723 \div 23\cdot934 = 4967\cdot23 \div 23934.$
 54. $\cdot793 \div \cdot49 = 79\cdot3 \div 49.$

55. $.001 \div 674.937 = 1 \div 674937$.
 56. Divide 47.655 by 4.5. *Ans.* 10.59.
 57. Divide 756.98 by 76.73612. *Ans.* 9.864+.
 58. Divide 47.5782975 by 26.175. *Ans.* 1.8177.
 59. Divide 1 by 7.6345. *Ans.* 0.1309+.
 60. Divide 75.347 by 0.3829. *Ans.* 196.7798+.
 61. Divide .0002 by .000000008. *Ans.* 25000.

 CONTRACTIONS IN DIVISION.

89. To Divide by 10, 100, 1000, &c.

Remove the decimal point as many places to the left in the dividend as there are 0s in the divisor.

90. To divide by 25.

Multiply by 4 and divide by 100.

Reason $25 = \frac{100}{4}$.

91. To divide by 15, 35, 45, or 55.

Double the dividend, and divide the product by 30, 70, 90, or 110, as the case may be.

REASON.—This method is simply doubling both the divisor and dividend. We must therefore divide the remainder, if any, by 2, for the *true* remainder.

92. To divide by 125.

Multiply the dividend by 8, and divide the product by 1000.

REASON.—This contraction is multiplying both the dividend and divisor by 8. For the *true* remainder, therefore, we must divide the remainder, if any, by 8.

93. To divide by 75, 175, 225, or 275.

Multiply the dividend by 4, and divide the product by 300, 700, 900, or 1100, as the case may be.

REASON.— $75 = \frac{300}{4}$, $175 = \frac{700}{4}$ &c. For the *true* remainder, divide the remainder, if any thus found, by 4.

94. When there are many decimals in the dividend and but few are required in the quotient, we may abbreviate the division by the following—

RULE.

Proceed as in Art. 71 till the decimal point is placed in the quotient, and then cut off a digit to the right hand of the divisor, at each new digit of the quotient; remembering to carry what would have been obtained by the multiplication of the digit neglected—unity if this multiplication would have produced more than 5, and less than 15; 2 if more than 15, and less than 25, &c.

EXAMPLE.—Divide 754·337385 by 61·347.

| Ordinary Method. | Contracted Method. |
|-----------------------|-----------------------|
| 61347)754337385(12296 | 61347)754337385(12296 |
| 61347 | 61347 |
| 140867 | 140867 |
| 122694 | 122694 |
| 181733 | 18173 |
| 122694 | 13269 |
| 590398 | 5904 |
| 552123 | 5521 |
| 382755 | 383 |
| 368082 | 368 |
| 146730 | 15 |

According as the denominations of the quotient become small, their products by the lower denomination of the divisor become inconsiderable, and may be neglected, and consequently, the portions of the dividend from which they would have been subtracted. What should have been *carried* from the multiplication of the digit neglected—since it belongs to a higher denomination than what is neglected—must still be retained.

EXERCISE.

61. The Ontario, Simcoe, and Huron Railway is 95 miles in length, and cost \$3300000. What was the cost per mile?
62. The Rideau Canal is 126 miles in length, and cost \$3860000. What was the average cost per mile?
63. The distance of the earth from the sun is 95270400 miles; how long would it take a cannon ball, going at the rate of 28800 miles per day, to reach the sun?
64. The national debt of France is 1145012096 dollars, and the number of inhabitants is 35781628; what is the amount of indebtedness of each individual?
65. The national debt of Great Britain is 3764112127 dollars, and the number of inhabitants is 27475271; what is the amount of indebtedness of each individual?
66. What is the ninth part of \$972?
67. What is each man's part, if \$972 be divided equally among 108 men?
68. Divide a legacy of \$8526 equally between 294 persons.
69. Divide 340480 ounces of bread equally between 792 persons.
70. A cubic foot of distilled water weighs 1000 ounces; what will be the weight of one cubic inch?
71. How many Sabbath days' journeys (each 1155 yards) in the Jewish day's journey, which was equal to 33 miles and 2 furlongs English?
72. How many pounds of butter, 19 cents per lb., would purchase a cow, the price of which is \$47·50?
73. Divide 978·634 by 96·34762.

74. Divide 729 bush. 1 pk. 1 gal. 1 qt. 1 pt. by 297.

75. Divide 179 cwt. 3 qr. 4 lb. 16 oz. by 9 lb. 7 oz. 8 drs.

76. The circumference of the earth is about 25000 miles; if a vessel sails 93 m. 4 fur. 7 rds. a day, how long will it require to sail round the earth?

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—*The numbers after the questions refer to the articles of the section.*

1. What is division? (58)
2. What is the divisor? (59)
3. What is the dividend? (60)
4. What is the quotient? What is the derivation of the word 'quotient'? (61)?
5. Explain when the quotient will be equal to unity, and when greater or less than unity? (62)
6. Under what circumstances does a remainder arise in division? (63)
7. What is the denomination of the remainder? (63)
8. Why can it never be as great as the divisor? (63)
9. What is the correspondence between the minuend and the subtrahend in subtraction and the divisor and the dividend in division? (64)
10. What may we consider as the factors of the dividend? (65)
11. How many ways have we of expressing the division of one quantity by another? What are they? (66)
12. When a quantity consisting of several terms, connected by the sign \times , is to be divided by any number, how may the work be performed? (67)
13. When a quantity consisting of several terms, connected by the signs $+$ or $-$, contained within brackets, is to be divided, what must be done upon removing the brackets? (68)
14. Give the general rule for division. (71)
15. In the question "Divide 11 m. 7 fur. 20 per. 3 yds. by 279," explain what is really required. (77) Show that all such questions are reducible to a species of subtraction. (72)
16. In dividing abstract numbers, explain what bringing down the next figure of the dividend is equivalent to. (73)
17. When there is a remainder, how is it to be written? (71, Example 1)
18. What are the three general principles upon which the operations of division depend? (76)
19. Why do we begin dividing at the left-hand side? (77)
20. How may division be proved? (78)
21. The divisor remaining unchanged, what effect has multiplying the dividend by any number? (79)
22. The divisor remaining unchanged, what effect has dividing the dividend by any number? (80)
23. The dividend remaining unchanged, what effect has dividing the divisor by any number? (81)
24. The dividend remaining unchanged, what effect has multiplying the divisor by any number? (82)
25. What is the effect upon the quotient when the divisor and the dividend are both multiplied by the same number? (83)
26. What is the effect upon the quotient when the divisor and the dividend are both divided by the same number? (84)
27. How do we divide by a composite number? (85)
28. When we divide by the divisors of a composite divisor, how do we obtain the correct remainder? (85)
29. When the divisor is separated into only two factors, how may the rule for obtaining the correct remainder be worded? (85)
30. When the divisor and the dividend are both denominate numbers, what is the rule? (86)
31. When one denominate number is divided by another, what kind of a number must the quotient always be? (87)

32. In the question "Divide 37 lb. 2 oz. 15 dr. by 1 lb. 9 oz. 11 dr.," what are we in reality required to do? (87)
33. When the divisor contains decimals, how do we proceed? (88) Upon what principle do we do this? (88)
34. How do we divide by 1, followed by any number of 0s? (89)
35. How do we contract the work when dividing by 25? How by 15, 35, 45, or 55? (90, 91)
36. How do we divide by 125? How by 75, 175, 225, or 275? (92, 93)
37. How do we abbreviate the work when there are many decimals in the dividend and but few are required in the quotient? (94)

MISCELLANEOUS EXERCISE.

(On preceding rules.)

1. Multiply 789643 by 999998.
2. Read the following numbers: 67813420·021030046,
72000000·000000072,
1001000100·0010000010000001.
3. Express 709, 4376, 9999, 86004, and 3947596 in Roman numerals.
4. Multiply 749 lb. 10 oz. avoirdupois by 72.
5. What is the price of 17 pairs of gloves at 4s. 7½d. per pair?
6. The planet Neptune is 2850 millions of miles from the sun; how long would it take a locomotive to travel from the sun to Neptune, at the rate of 30 miles an hour?
7. Reduce £729 17s. 6½d. to dollars and cents.
8. From \$10000 subtract \$9876·23.
9. Write down five hundred and twenty billions, six millions, two thousand and forty-three, and five thousand and sixteen trillionths.
10. Reduce 7964327 inches to acres, roods, &c.
11. Add together the following quantities: \$729·43, \$16·70, \$976·81, \$9987·17, \$429·00, \$129·19.
12. Multiply 6 weeks 4 days 3 hours 17 minutes by 429.
13. Take the number 741, and, by removing the decimal point: (1) multiply it by 1000000; (2) divide it by 100000; (3) make it millions; (4) make it billionths; (5) make it trillionths; (6) make it hundredths of thousandths; (7) make it tenths.
14. Multiply 78·96 by ·00042.
15. How many hogsheads of sugar, each containing 13 cwt. 2 qrs. 14 lbs., may be put on board a ship of 324 tons burden?
16. A farmer's yearly income was 9237 dollars. He paid for repairing his house 136 dollars, for hired help on his farm 4 times as much lacking 95 dollars, and for other expenses 1902 dollars; how much does he save yearly?
17. How many suits of clothes can be made from a piece of cloth containing 39 yds. 2 qrs. 3 nls.; each suit requiring 3 yds. 1 qr. 2 nls.?
18. There is a farm consisting of 732 acres; 25 acres of which is planted with corn and potatoes; 197 acres sown with rye; 156 with oats; 97 with wheat; 199 is pastured; and the remainder is meadow. How many acres of meadow?

19. Bought 96 acres 3 roods 17 perches of land, for which I pay \$7764; what did I pay for it per perch?

20. A lady, having 312 dollars, paid for a bonnet 20 dollars, for a shawl 75 dollars, for a silk dress 97 dollars, and for some delaines 83 dollars; how much had she remaining?

21. A silversmith received 36 lb. 8 oz. 14 dwt. 16 grs. of silver to make 12 tankards; what would the weight of each tankard be?

22. I bought four fields; in the first there were 6 acres 3 rds. 12 perches; in the second, 7 acres 2 roods; in the third, 9 acres and 13 perches; in the fourth, 5 acres 2 roods 36 perches. How much in all?

23. A merchant expended 294 dollars for broadcloth, consisting of three different kinds; the first at 5 dollars a yard; the second at 7 dollars; and the third at 9 dollars a yard. He had as many yards of one kind as of another—how many yards of each kind did he buy?

24. A silversmith made three dozen spoons, weighing 5 lb. 9 oz. 8 dwt.; a tea-pot, weighing 3 lb. 2 oz. 16 dwt. 16 grs.; two pair of silver candlesticks, weighing 4 lb. 6 oz. 17 dwt.; a dozen silver forks, weighing 1 lb. 8 oz. 19 dwt. 22 grs.: what was the weight of all the articles?

25. Reduce £972 11s. 11½d. to dollars and cents.

26. Reduce 179 lbs. 3 oz. 3 dr. 1 scr. 14 grs. to grains.

27. There is a house 56 feet long, and each of the two sides of the roof is 25 feet wide; how many shingles will it take to cover it, if it require 6 shingles to cover a square foot?

28. A merchant bought 4 bales of cotton; the first contained 6 cwt. 2 qr. 11 lb.; the second, 5 cwt. 3 qr. 16 lb.; the third, 8 cwt. 0 qr. 7 lb.; the fourth, 3 cwt. 1 qr. 17 lb. He sold the whole at 15 cents a pound; what did it amount to?

29. A merchant has 29 bales of cotton cloth, each bale containing 57 yards; what is the value of the whole at 15 cents a yard?

30. A man willed an estate of \$370129 to his two children and wife, as follows: to his son, \$139468; to his daughter, \$98579; and to his wife the remainder. How much did he will to his wife?

31. Divide £1694 16s. 0½d. by £9 19s. 11½d.

32. Reduce £19 19s. 11½d. to dollars and cents.

33. A merchant having purchased 12 cwt. of sugar, sold at one time 3 cwt. 2 qrs. 11 lb., and at another time he sold 4 cwt. 1 qr. 15 lb.; what is the remainder worth, at 15 cents per pound?

34. Bought 4 chests of hyson tea; the weight of the first was 2 cwt. 0 qr. 17 lb.; the second, 3 cwt. 2 qrs. 15 lb.; the third, 2 cwt. 1 qr. 20 lb.; the fourth, 5 cwt. 3 qrs. 17 lb.; what is the value of the whole at 37½ cents a pound?

35. Express 100200300709 in Roman numerals.

36. Divide 43.2 by 76.8437.
37. Divide 123.4 by .000000066.
38. From \$2789.27 take 17 times \$63.29.
39. Add together \$278.43, \$417.16, \$11.27, \$2110.40, \$723.15, and £29 6s. 11½d. and divide the sum by 173.
40. In 1857 the total number of volumes in the Common School and other Public Libraries of Canada West was estimated at 491544 and the number of libraries at 2076. How many volumes were there upon an average to each library?

SECTION III.

PROPERTIES OF NUMBERS, PRIME NUMBERS, MEASURES,
GREATEST COMMON MEASURE, LEAST COMMON
MULTIPLE, SCALES OF NOTATION, AND APPLICA-
TION OF THE FUNDAMENTAL RULES TO DIFFERENT
SCALES. DUODECIMALS.

1. A divisor, or measure of a number, is a number which will divide it exactly; that is, leaving no remainder.
2. A multiple of a number is a number of which the given number is a divisor.
3. An integer, or integral number, is a whole number.
4. Integers are either *prime* or *composite*, *odd* or *even*.
5. An Even Number is that of which 2 is a divisor.
6. An Odd Number is that of which 2 is not a divisor.
7. A Prime Number is one which has no integral divisor except unity and itself, thus 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, &c., are primes.
8. A Composite Number is a number which is not prime; or is a number which has other *integral* divisors besides unity and itself, thus 4, 6, 9, 10, 12, 14, 15, 16, 21, &c., are composite numbers.
9. The Factors of a number are those numbers which, when multiplied together, *produce* or *make* it.
10. Factors are sometimes called measures, submultiples, or aliquot parts.
11. A Common Measure of two or more numbers, is a number which will divide each of them without a remainder; thus 7 is a common measure of 14, 35, and 63.
12. Two or more numbers are prime to one another when they have no common divisor except unity; thus, 9 and 14 are "prime to each other."

Hence all prime numbers are prime to each other; but composite numbers may or may not be prime to one another.

13. Commensurable Numbers are those which have some common divisor.

Thus 55 and 33 are commensurable, the common divisor being 11.

14. Incommensurable Numbers are those which are prime to one another.

Thus 55 and 34 are incommensurable.

15. A Square Number is one which is composed of two equal factors.

Thus $25=5 \times 5$ is a square number; so also $64=8 \times 8$, &c.

16. A Cube Number is one which is composed of three equal factors.

Thus $343=7 \times 7 \times 7$ is a cube number; so also $27=3 \times 3 \times 3$, &c.

17. A Perfect Number is one which is exactly equal to the sum of all its divisors.

Thus, $6=1+2+3$ is a perfect number; so also $28=1+2+4+7+14$ is a perfect number.

All the numbers known to which this property really belongs, are the eight following: 6; 28; 496; 8128; 33550336; 8589869056; 137438691328; and 2305843008139962128.

NOTE.—All perfect numbers terminate with 6, or 28.

18. Amicable Numbers are such pairs of integers that each of them is exactly equal to the sum of all the divisors of the other.

Thus, 220 and 284 are amicable; for, $220=1+2+4+7+14$, which are all the divisors of 284 and $284=1+2+5+11+4+10+22+20+44+55+110$, which are all divisors of 220.

Other amicable numbers are 17296 and 18416; also 9363583 and 9437056.

19. By the term *properties of numbers*, is meant those qualities or elements which are inseparable from them. Some of the most important properties of numbers are the following:

I. The sum of two or more even numbers is an even number.

II. The difference of two even numbers is an even number.

III. The sum or difference of two odd numbers is an even number.

IV. The sum of three, five, seven, &c., odd numbers, is an odd number.

V. The sum of two, four, six, eight, &c., odd numbers, is an even number.

VI. The sum or difference of an even and an odd number, is an odd number.

VII. The product of two even numbers, or of an even and an odd number, is an even number.

VIII. If an even number be divisible by an odd number, the quotient will be an even number.

IX. The product of any number of factors will be even if one of the factors be even.

X. An odd number is not divisible by any even number.

XI. The product of any number of factors is odd if they are all odd.

XII. If an odd number divide an even number, it will also divide half of it.

XIII. Any number that measures two others must likewise measure their *sum*, their *difference*, and their *product*.

Thus, if 6 goes into 24 four times, and into 18 three times, it will go into 24+18 or 42, three plus four, or seven times.

Also, if 6 goes into 24 four times, and into 42 seven times, it will go into 42-24 or 18, seven minus four, or three times.

Lastly, if 6 goes into 24 four times, and into 12 twice, it will evidently go into 12 times 24, twelve times 4 times, or 48 times.

XIV. If one number measure another, it must likewise measure any multiple of that other.

Thus, if 7 measures 21, it must evidently measure 6 times 21, or 11 times 21, or 17 times 21, &c.

XV. Any number, expressed by the decimal notation, divided by 9, will leave the same remainder as the sum of its digits divided by 9. (See Art. 55, Sec. II.)

This property of the number 9 affords an ingenious method of proving each of the fundamental rules. The same property belongs to the number 3; for 3 is a measure of 9, and will therefore be contained an exact number of times in any number of 9s. But it belongs to no other digit.

The preceding is not a *necessary* but an *incidental* property of the number 9. It arises from the *law of increase* in the decimal notation. If the *radix* of the system were 8, it would belong to 7; if the radix were 12, it would belong to 11; and, universally, it belongs to the number that is *one less* than the *radix* of the system of notation.

XVI. If the number 9 be multiplied by any *single* digit, the *sum* of the figures composing the product will make 9.

Thus, $9 \times 4 = 36$, and $3+6=9$; so also $8 \times 9 = 72$ and $7+2=9$.

XVII. If we take any two numbers whatever; then *one* of them, or their *sum*, or their *difference*, is divisible by 3.

Thus, take 11 and 17; though neither the numbers themselves, nor their sum, is divisible by 3, yet their difference is, for it is 6.

XVIII. Any number divided by 11, will leave the *same remainder* as the sum of its *alternate* digits in the *even* places, reckoning from the right, taken from the sum of its alternate digits in the *odd* places, increased by 11, if necessary.

Take any number, as 38405603, and mark the alternate figures. Now the sum of those marked, viz: $8+0+6+3=17$. The sum of others, viz: $3+4+5+0=12$. And $17-12=5$, the remainder sought. That is, 38405603 divided by 11, will leave 5 remainder.

Again, take 5847362, the sum of the marked figures is 14; the sum of those not marked is 21. Now 21 taken from 25, (*i. e.* 14 increased by 11,) leaves 4. the remainder sought=remainder obtained by dividing 5847362 by 11.

XIX. Any number ending in 0, or an even number, is divisible by 2.

XX. Any number ending in 5 or 0 is divisible by 5.

XXI. Any number ending in 0 is divisible by 10.

XXII. When the two right-hand figures are divisible by 4, the whole is divisible by 4.

XXIII. When the three right-hand figures are divisible by 8, the whole number is divisible by 8.

XXIV. When the sum of the digits of a number is divisible by 9, the number itself is divisible by 9.

XXV. When the sum of the digits of a number is divisible by 3, the number itself is divisible by 3.

XXVI. When the sum of the digits, standing in the *even* places, is equal to the sum of the digits standing in the *odd* places, the number is divisible by 11.

Thus, to illustrate the last five properties,

The number 7416 is divisible by 4, because 16, the last two digits, are divisible by 4.

— is divisible by 8, because 416, its last three digits, are divisible by 8.

— is divisible by 9, because the sum of its digits, $7+4+1+6=18$, is divisible by 9.

— is divisible by 3, because the sum of its digits, $7+4+1+6=18$, is divisible by 3.

So also the number 4567321 is divisible by 11, since the sum of the digits in the odd places, $1+3+6+4=14=2+7+5$, the sum of the digits in the even places.

XXVII. Every *composite number* may be resolved into *prime factors*.

For, since a composite number is produced by multiplying two or more factors together, it may evidently be resolved into those factors; and if these factors themselves are *composite*, they also may be resolved into *other factors*, and thus the analysis may be continued until all the factors are *prime numbers*.

XXVIII. The *least* divisor of any number is a prime number.

For every whole number is either prime or composite (Art. 4); but a composite number can be resolved into factors (XXVII): consequently, the *least* divisor of any number must be a *prime* number.

XXIX. Every prime number, except 2, if increased or diminished by 1, is divisible by 4. (See table of prime numbers on next page).

XXX. Every prime number, except 2, is odd; and therefore terminates in an odd digit.

NOTE.—It must not be inferred from this that all *odd* numbers are *prime*.

XXXI. All prime numbers, except 2 and 5, must terminate with 1, 3, 7, or 9. Every number that ends in any other digit than 1, 3, 7, or 9, is a composite number.

For all prime numbers, except 2, must end in an odd digit (XXIX), and all numbers ending in 5 are divisible by 5.

XXXII. Every prime number, except 2 and 3, if increased or diminished by 1, is divisible by 6.

20. To find the prime numbers between any given limits—

RULE.

Write down all the odd numbers, 1, 3, 5, 7, 9, &c. Over every third from 3 write 3; over every fifth from 5 write 5; over every seventh from 7 write 7; over every eleventh from 11 write 11; and so on.

Then all the numbers which are thus marked are composite; and the others, together with 2, are prime.

Also the figures thus placed over, are factors of the numbers over which they stand.

EXAMPLE.

Find all the prime numbers less than 100.

| | | | | | | | | |
|-----------------------|-----------------------|-----------------------|---------|-----------------------|-----------------------|-----------------------|-----------------------|----------------------|
| 1 | 3 | 5 | 7 | 3 | 11 | 13 | 3 ⁵ 15 | 17 |
| | 3 ⁷ 21 | | 5 25 | 3 27 | | | 3 ¹¹ 33 | 5 ⁷ 35 |
| | 3 ¹³ 39 | 41 | 43 | 3 ⁵ 45 | 47 | 7 49 | 3 ¹⁷ 51 | 53 |
| 5 ¹¹ | 3 ¹⁹ 57 | 59 | 61 | 3 ⁷ 63 | 5 ¹³ 65 | 67 | 3 ²³ 69 | 71 |
| 73 | 3 ⁵ 75 | 7 ¹¹ 77 | 79 | 3 81 | 83 | 5 ¹⁷ 85 | 3 ²⁹ 87 | 89 |
| 7 ¹³ 91 | 3 ³¹ 93 | 5 ¹⁹ 95 | 97 | 3 ¹¹ 99 | | | | |

Hence, rejecting all the numbers which have *superiors*, the primes less than 100 are 1, 3, 5, 7, 11, 13, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, together with the number 2.

This process may be extended indefinitely, and is the method by which primes are found even by modern computers. It was invented by Eratosthenes, a learned librarian at Alexandria (Born B. C. 275). He inscribed the series of odd numbers upon parchment, then cutting out such numbers as he found to be composite, his parchment with its holes somewhat resembled a *sieve*: hence, this method is called '*Eratosthenes' Sieve*.'

TABLE OF PRIME NUMBERS FROM 1 TO 3407.

| | | | | | | | | | | | |
|-----|-----|-----|-----|------|------|------|------|------|------|------|------|
| 1 | 173 | 409 | 659 | 941 | 1223 | 1511 | 1811 | 2129 | 2423 | 2741 | 3079 |
| 2 | 179 | 419 | 661 | 947 | 1229 | 1523 | 1823 | 2131 | 2437 | 2749 | 3083 |
| 3 | 181 | 421 | 673 | 953 | 1231 | 1531 | 1831 | 2137 | 2441 | 2753 | 3089 |
| 5 | 191 | 431 | 677 | 967 | 1237 | 1543 | 1847 | 2141 | 2447 | 2757 | 3109 |
| 7 | 193 | 433 | 683 | 971 | 1249 | 1549 | 1861 | 2143 | 2459 | 2777 | 3119 |
| 11 | 197 | 439 | 691 | 977 | 1259 | 1553 | 1867 | 2153 | 2467 | 2789 | 3121 |
| 13 | 199 | 443 | 701 | 983 | 1277 | 1559 | 1871 | 2161 | 2473 | 2791 | 3137 |
| 17 | 211 | 449 | 709 | 991 | 1279 | 1567 | 1873 | 2179 | 2477 | 2797 | 3163 |
| 19 | 223 | 457 | 719 | 997 | 1283 | 1571 | 1877 | 2203 | 2503 | 2801 | 3167 |
| 23 | 227 | 461 | 727 | 1009 | 1289 | 1579 | 1879 | 2207 | 2521 | 2803 | 3169 |
| 29 | 229 | 463 | 733 | 1013 | 1291 | 1583 | 1889 | 2213 | 2531 | 2819 | 3181 |
| 31 | 233 | 467 | 739 | 1019 | 1297 | 1597 | 1901 | 2221 | 2539 | 2833 | 3187 |
| 37 | 239 | 479 | 743 | 1021 | 1301 | 1601 | 1907 | 2237 | 2543 | 2837 | 3191 |
| 41 | 241 | 487 | 751 | 1031 | 1303 | 1607 | 1913 | 2239 | 2549 | 2843 | 3203 |
| 43 | 251 | 491 | 757 | 1033 | 1307 | 1609 | 1931 | 2243 | 2551 | 2851 | 3209 |
| 47 | 257 | 499 | 761 | 1039 | 1319 | 1613 | 1933 | 2251 | 2557 | 2857 | 3217 |
| 53 | 263 | 503 | 769 | 1049 | 1321 | 1619 | 1949 | 2267 | 2579 | 2861 | 3221 |
| 59 | 269 | 509 | 773 | 1051 | 1327 | 1621 | 1951 | 2269 | 2591 | 2879 | 3229 |
| 61 | 271 | 521 | 787 | 1061 | 1361 | 1627 | 1973 | 2273 | 2593 | 2887 | 3251 |
| 67 | 277 | 523 | 797 | 1063 | 1367 | 1637 | 1979 | 2281 | 2609 | 2897 | 3253 |
| 71 | 281 | 541 | 809 | 1069 | 1373 | 1657 | 1987 | 2287 | 2617 | 2903 | 3257 |
| 73 | 283 | 547 | 811 | 1087 | 1381 | 1663 | 1993 | 2293 | 2621 | 2909 | 3259 |
| 79 | 293 | 557 | 821 | 1091 | 1399 | 1667 | 1997 | 2297 | 2633 | 2917 | 3271 |
| 83 | 307 | 563 | 823 | 1093 | 1409 | 1669 | 1999 | 2309 | 2647 | 2927 | 3299 |
| 89 | 311 | 569 | 827 | 1097 | 1423 | 1693 | 2003 | 2311 | 2657 | 2939 | 3301 |
| 97 | 313 | 571 | 829 | 1103 | 1427 | 1697 | 2011 | 2333 | 2659 | 2953 | 3307 |
| 101 | 317 | 577 | 839 | 1109 | 1429 | 1699 | 2017 | 2339 | 2663 | 2957 | 3313 |
| 103 | 331 | 587 | 863 | 1117 | 1433 | 1709 | 2027 | 2341 | 2671 | 2963 | 3319 |
| 107 | 337 | 593 | 867 | 1123 | 1439 | 1721 | 2029 | 2347 | 2677 | 2969 | 3323 |
| 109 | 347 | 599 | 869 | 1129 | 1447 | 1723 | 2039 | 2351 | 2683 | 2971 | 3329 |
| 113 | 349 | 601 | 863 | 1151 | 1451 | 1733 | 2053 | 2357 | 2687 | 2999 | 3331 |
| 127 | 353 | 607 | 877 | 1153 | 1453 | 1741 | 2063 | 2371 | 2689 | 3001 | 3343 |
| 131 | 359 | 613 | 881 | 1163 | 1459 | 1747 | 2069 | 2377 | 2693 | 3011 | 3347 |
| 137 | 367 | 617 | 883 | 1171 | 1471 | 1753 | 2081 | 2381 | 2699 | 3019 | 3359 |
| 139 | 373 | 619 | 887 | 1181 | 1481 | 1759 | 2083 | 2383 | 2707 | 3023 | 3361 |
| 149 | 379 | 631 | 907 | 1187 | 1483 | 1777 | 2087 | 2389 | 2711 | 3037 | 3371 |
| 151 | 383 | 641 | 911 | 1193 | 1487 | 1783 | 2089 | 2393 | 2713 | 3041 | 3373 |
| 157 | 389 | 643 | 919 | 1201 | 1489 | 1787 | 2099 | 2399 | 2719 | 3049 | 3389 |
| 163 | 397 | 647 | 929 | 1213 | 1493 | 1789 | 2111 | 2411 | 2729 | 3061 | 3391 |
| 167 | 401 | 653 | 937 | 1217 | 1499 | 1801 | 2113 | 2417 | 2731 | 3067 | 3407 |

When it is required to determine whether a given number is a prime, we first notice the terminating figure; if it is different from 1, 3, 7, or 9, the number is composite; but if it terminate with one of the above digits, we must endeavour to divide it with some one of the primes, as found in the table, commencing with 3. There is no necessity for trying 2, for 2 will divide only the even numbers. If we proceed to try all the successive primes of the table until we reach a prime which is not less than the square-root

of the number, without finding a divisor, we may conclude with certainty that the number is a *prime*.

The reason why we need not try any primes greater than the square-root of the number, is drawn from the following consideration: If a composite number is resolved into two factors, one of which is less than the square-root of the number, the other must be greater than the square-root.

The square of the last prime given in our table is 11607649; hence, this table is sufficiently extended to enable us to determine whether any number not exceeding 11607649 is a prime. It is obvious that numbers may be proposed which would require by this method very great labor to determine whether they are primes, still this is the only sure and general method as yet discovered.

21. TO RESOLVE A COMPOSITE NUMBER INTO ITS PRIME FACTORS.

RULE.

Divide the given number by the smallest number which will divide it without a remainder; then divide the quotient in the same way, and thus continue the operation till a quotient is obtained which can be divided by no number greater than 1. The several divisors with the last quotient, will be the prime factors required. (19-XXVII.)

REASON.—Every division of a number, it is plain, resolves it into two factors, viz. the divisor and the quotient. But according to the rule, the divisors, in every case, are the *smallest* numbers that will divide the given number or the successive quotients without a remainder, consequently they are all *prime* numbers. (19-XXVIII.) And since the division is continued till a quotient is obtained, which cannot be divided by any number but unity or itself, it follows that the *last* quotient must also be a *prime* number; for, a prime number is one which cannot be exactly divided by any whole number except *unity* and *itself*. (Art. 7.)

NOTE.—Since the *least divisor* of every number is a *prime* number, it is evident that a composite number may be resolved into its prime factors by dividing it continually by *any prime number* that will divide the given number and the successive quotients without a remainder. Hence,

A composite number can be divided by any of its *prime factors* without a remainder, and by the product of any two or more of them, but by *no other number*.

Thus, the prime factors of 42 are 2, 3, and 7. Now 42 can be divided by 2, 3, and 7; also 2×3 , 2×7 , 3×7 , and $2 \times 3 \times 7$; but it can be divided by no other number.

EXAMPLE 1.—Resolve 210 into its prime factors.

OPERATION. We first divide the given number by 2, which is the least number that will divide it without a remainder, and which is also a prime number. We next divide by 3, then by 5. The several divisors and the last quotient are the prime factors required.

5)37

7

PROOF.— $2 \times 3 \times 5 \times 7 = 210$.

Ans. 2, 3, 5, and 7.

EXAMPLE 2.—Resolve 728 into its prime factors.

OPERATION.

2)728

2)364

2)182

7) 91

13

Therefore, $2 \times 2 \times 2 \times 7 \times 13$ or $2^3 \times 7 \times 13$, are the prime factors of 728.

EXERCISE.

- | | |
|--|---|
| 3. Resolve 11368 into its prime factors. | <i>Ans.*</i> $2^3 \times 7^2 \times 29$. |
| 4. What are the prime factors of 2934 ? | <i>Ans.</i> $2 \times 3^2 \times 163$. |
| 5. What are the prime factors of 1011 ? | <i>Ans.</i> 3×337 . |
| 6. What are the prime factors of 1000 ? | <i>Ans.</i> $2^3 \times 5^3$. |
| 7. What are the prime factors of 1024 ? | <i>Ans.</i> 2^{10} . |
| 8. What are the prime factors of 32320 ? | <i>Ans.</i> $2^6 \times 5 \times 101$. |
| 9. What are the prime factors of 707 ? | <i>Ans.</i> 7×101 . |
| 10. What are the prime factors of 1118 ? | <i>Ans.</i> $2 \times 13 \times 43$. |

DIVISORS.

22. From Art. 21, Note, for finding all the divisors of any number, we deduce the following—

RULE.

Resolve the number into its prime factors; form as many series of terms as there are prime factors, by making 1 the first term of each series, the first power of one of the prime factors for the second term, the second power of this factor for the third term, and so on, until we reach the highest that occurred in the decomposition. Then multiply these series together, and the partial products thus obtained will be the divisor sought.

EXAMPLE 1.—What are the divisors of 48 ?

Here we find $48=2^4 \times 3$. Therefore our series of terms will be $1 \cdot 2 \cdot 4 \cdot 8 \cdot 16$ and $1 \cdot 3$; multiplying these together.

$$\begin{array}{r} 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \\ 1 \cdot 3 \\ \hline 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 3 \cdot 6 \cdot 12 \cdot 24 \cdot 48 \end{array}$$

Therefore the divisors of 48 are 1, 2, 3, 4, 6, 8, 12, 16, 24, and 48.

We begin each series with 1, because, were we not to do so, the different powers of the prime factors would not themselves appear among the partial products.

EXAMPLE 2.—What are the divisors of 360.

The prime factors of 360 are $2^3 \times 3^2 \times 5$ and therefore the series are $1 \cdot 2 \cdot 4 \cdot 8$; $1 \cdot 3 \cdot 9$ and $1 \cdot 5$.

OPERATION.

$$\begin{array}{r} 1 \cdot 2 \cdot 4 \cdot 8 \\ 1 \cdot 3 \cdot 9 \\ \hline 1 \cdot 2 \cdot 4 \cdot 8 \cdot 3 \cdot 6 \cdot 12 \cdot 24 \cdot 9 \cdot 18 \cdot 36 \cdot 72 = \text{product of 1st and 2nd series} \\ 1 \cdot 5 \\ \hline 1 \cdot 2 \cdot 4 \cdot 8 \cdot 3 \cdot 6 \cdot 12 \cdot 24 \cdot 9 \cdot 18 \cdot 36 \cdot 72 \cdot 5 \cdot 10 \cdot 20 \cdot 40 \cdot 15 \cdot 30 \cdot 60 \cdot \\ 120 \cdot 45 \cdot 90 \cdot 180 \cdot 360. \\ \text{Therefore the divisors of 360 are 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30,} \\ 36, 40, 45, 60, 72, 90, 120, 180, 360. \end{array}$$

* The small figures written to the right of the factors and above the line are called exponents, and show how often the digit is taken as factor.

EXERCISE.

3. What are the divisors of 100 ?
Ans. 1, 2, 4, 5, 10, 20, 25, 50, 100.
4. What are the divisors of 810 ?
Ans. { 1, 2, 3, 5, 6, 9, 10, 15, 18, 27, 30, 45, 54, 81, 90, 135, 162,
 270, 405, 810.
5. What are the divisors of 920 ?
Ans. 1, 2, 4, 5, 8, 10, 20, 23, 40, 46, 92, 115, 184, 230, 460, 920.
6. What are the divisors of 25000 ?
Ans. { 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 125, 200, 250, 500,
 625, 1000, 1250, 2500, 3125, 5000, 6250, 12500, 25000.

NUMBER OF DIVISORS.

23. Since the series of terms which we multiplied together, by the last rule, to obtain the divisors of any number commenced with 1, it follows that the number of terms in each series will be one more than the units in the exponent of the factor used.

Hence, to find the *number* of divisors of any number, without actually setting them down, we have the following—

RULE.

Resolve the number into its prime factors and express them as in example 3, 4, and 6 in Art. 21. Increase each exponent by unity and multiply the resulting numbers together. The product will be the number of divisors.

EXAMPLE 1.—How many divisors has 4320 ?

$4320 = 2^6 \times 3^3 \times 5$. Here the exponents are 6, 3, and 1: each of which being increased by one, we obtain 7, 4, and 2, the continued product of which is $7 \times 4 \times 2 = 56$ —the number of divisors sought.

EXERCISE.

- | | |
|----------------------------------|------------------|
| 2. How many divisors has 88200 ? | <i>Ans.</i> 108. |
| 3. How many divisors has 3500 ? | <i>Ans.</i> 24. |
| 4. How many divisors has 6336 ? | <i>Ans.</i> 42. |
| 5. How many divisors has 824 ? | <i>Ans.</i> 8. |
| 6. How many divisors has 49000 ? | <i>Ans.</i> 48. |
| 7. How many divisors has 81000 ? | <i>Ans.</i> 80. |
| 8. How many divisors has 75600 ? | <i>Ans.</i> 120. |
| 9. How many divisors has 25600 ? | <i>Ans.</i> 33. |

GREATEST COMMON MEASURE.

24. The greatest common measure, or greatest common divisor of two or more numbers, is the greatest number that will divide each of them without a remainder.

25. To find a common divisor or common measure of two or more numbers:—

RULE.

Resolve the given numbers into their prime factors, then, if any factor be common to all, it will be a common measure.

If the given numbers have not a common factor, they cannot have a common measure greater than unity, and consequently are either prime numbers or are prime to each other. (Arts. 7 and 12.)

EXAMPLE 1.—Find a common divisor of 14, 35, and 63.

$14 = 2 \times 7$; $35 = 5 \times 7$, and $63 = 3 \times 3 \times 7$. The factor 7 is common to all the given numbers, and is therefore a common measure of them.

EXERCISE.

2. Find a common divisor of 21, 18, 27, and 36. *Ans.* 3.
3. Find a common divisor of 21, 77, 42, and 35. *Ans.* 7.
4. Find a common divisor of 26, 52, 91, and 143. *Ans.* 13.
5. Find a common divisor of 82, 118, and 146. *Ans.* 2.

26. To find the greatest common measure of two quantities:—

RULE.

Divide the larger by the smaller; then the divisor by the remainder; next the preceding divisor by the new remainder:—continue this process until nothing remains, and the last divisor will be the greatest common measure. If this be unity, the given numbers are prime to each other.

EXAMPLE 1.—Find the greatest common measure of 3252 and 4248.

$$\begin{array}{r}
 3252)4248(1 \\
 \underline{3252} \\
 996)3252(3 \\
 \underline{2988} \\
 264)996(3 \\
 \underline{792} \\
 204)264(1 \\
 \underline{204} \\
 60)204(3 \\
 \underline{180} \\
 24)60(2 \\
 \underline{48} \\
 12)24(2 \\
 \underline{24}
 \end{array}$$

996, the first remainder, becomes the second divisor; 264, the second remainder, becomes the third divisor, &c. 12, the last divisor, is the required greatest common measure.

PROOF.—In order to establish the truth of this rule, it is necessary to remember (19-XIII. and XIV.) that if one number measure another it will likewise measure any integral multiple of that other; and if one number measure two others, it will also measure their sum or their difference.

First, then, 12 is a common measure of 3252 and 4248. Beginning at the end of the process: because 12 measures 12, it also measures 24, a multiple of 12; because 12 measures 24, it measures 48, a multiple of 24; because 12 measures 12 and also 48, it measures 60, which is their sum; because 12 measures 60, it measures 180, a multiple of 60; because 12 measures 180, and also 24, it measures their sum, which is 204; because 12 measures 204, and likewise 60, it measures their sum, 264; because 12 measures 264, it measures 792, a multiple of 264; and because 12 measures 792, and also 204, it measures their sum, which is 996; because 12 measures 996, it measures 2988, a multiple of 996; and because 12 measures 2988, and also 264, it measures their sum, 3252; and because 12 measures 3252, and also 996, it measures their sum, which is 4248. 12, therefore, measures each of the given numbers, and is a common measure; next it is their *greatest* common measure.

For, if not, let some other, as 13, be greater. Then, (beginning now at the top of the process) because 13 measures 3252, and also 4248, it measures their difference, which is 996; because 13 measures 996, it measures 2988, a multiple of 996, and because 13 measures 3252, and also 2988, it also measures their difference, which is 264; because 13 measures 264, it also measures 792, a multiple of 264; and because 13 measures 792, and also 996, it measures their difference, which is 204; because 13 measures 204, and also 204, it measures their difference, which is 60; because 13 measures 60, it measures 180, a multiple of 60; and because 13 measures 180, and also 204, it measures their difference, which is 24; because 13 measures 24, it measures 48, a multiple of 24; and because 13 measures 60, and also 48, it measures their difference, which is 12. That is, 13 measures or divides 12—a greater number measures a less, which is impossible.

Therefore 13 is not a common measure of 3252 and 4248; and in a similar manner it may be shown that no number greater than 12 is a common measure. Therefore 12 is the greatest common measure.

As the rule might be proved for any other example equally well, it is true in all cases.

EXERCISE.

2. What is the greatest common measure of 296 and 407?
Ans. 37.
3. What is the greatest common measure of 506 and 308?
Ans. 22.
4. What is the greatest common measure of 74 and 84? *Ans.* 2.
5. What is the greatest common measure of 1825 and 2555?
Ans. 365.
6. What is the greatest common measure of 556 and 672?
Ans. 4.

27. To find the greatest common measure of more than two numbers:—

RULE.

Find the greatest common measure of two of them; then, of this common measure and a third; next, of this last common measure and a fourth, &c. The last common measure found will be the greatest common measure of all the given numbers.

EXAMPLE 7.—Find the greatest common measure, of 679, 5901, and 6734.

By the last rule we find that 7 is the greatest common measure of 679 and 5901; and by the same rule, that it is the greatest common measure of 7 and 6734 (the remaining number), for $6734 \div 7 = 962$, with no remainder. Therefore 7 is the required number.

EXAMPLE 8.—Find the greatest common measure of 936, 736, and 142.

The greatest common measure of 936 and 736 is 8, and the greatest common measure of 8 and 142 is 2; therefore 2 is the greatest common measure of the given numbers.

This rule may be shown to be correct in the same way as the last; except that in proving the number found to be a *common* measure, we are to begin at the end of *all* the processes, and go through all of them in succession; and in proving that it is the *greatest* common measure, we are to begin at the commencement of the first process, or that used to find the common measure of the two first numbers, and proceed successively through *all*.

EXERCISE.

9. What is the greatest common measure of 110, 140, and 680?
Ans. 10.
10. What is the greatest common measure of 1326, 3094, and 4420?
Ans. 442.
11. What is the greatest common measure of 468, 922, and 375?
Ans. They have none.
12. What is the greatest common measure of 204, 1190, 1445, and 2006?
Ans. 17.

SECOND METHOD.

28. It is manifest that the greatest common measure or greatest common divisor of two or more numbers, must be their greatest common factor, and that this greatest common factor must be the product of all the prime factors that are common to all the given numbers.

Hence, to find the greatest common measure of two or more numbers, we have the following:—

RULE.

Resolve each of the given numbers into its prime factors; and the product of those factors, which are common to all, will be the greatest common measure.

EXAMPLE 13.—What is the greatest common measure of 1365 and 1995.

$$\begin{array}{r} 3)1365 \\ \hline 5)455 \\ \hline 7)91 \\ \hline 13 \\ \hline \end{array}$$

Hence, 3, 5, 7, and 13 are the prime factors.

$$\begin{array}{r} 3)1995 \\ \hline 5)665 \\ \hline 7)133 \\ \hline 19 \\ \hline \end{array}$$

Hence, 3, 5, 7, and 19 are the prime factors.

And the factors that are common to both are 3, 5, 7. Hence $3 \times 5 \times 7 = 105$ = greatest common measure.

EXAMPLE 14.—What is the greatest common measure of 108, 126, and 162 ?

$108 = 2^2 \times 3^3$, $126 = 2 \times 3^2 \times 7$, and $162 = 2 \times 3^4$.

Hence, the factors that are common are 2 and 3^2 , and the greatest common measure $= 2 \times 3^2 = 18$.

EXERCISE.

Work by this method all the preceding examples.

15. What is the greatest common measure of 56, 84, 140, 168 ?
Ans. 28.
16. What is the greatest common measure of 241920, 380160, 69120, 103680 ?
Ans. 34560.
17. What is the greatest common measure of 10800, 28040, and 2160 ?
Ans. 40.

LEAST COMMON MULTIPLE.

29. One number is a common multiple of two or more others when it can be divided by each of them without a remainder.

30. One number is the least common multiple (l. c. m.) of two or more others when it is the *least* number that can be divided by each of them without a remainder.

31. It is evident that a dividend will contain a divisor an exact number of times, when it contains, as factors, *every* factor of that divisor; and hence, the question of finding the *least* common multiple of several numbers is reduced to finding a number which shall contain all the prime factors of each number and *none others*. If the numbers have no common prime factor, their product will be their least common multiple.

Suppose we wish to see what is the least common multiple of 9, 12, 16, 20, and 35. Resolving these into their prime factors, we obtain $9 = 3^2$, $12 = 2^2 \times 3$, $16 = 2^4$, $20 = 2^2 \times 5$, and $35 = 7 \times 5$. Now it is plain that 2^4 must enter into the least common multiple as a factor, and, since 2^4 is a multiple of 2^2 , we do not consider 2^2 also a factor of the least common multiple. So also 3^2 must be a factor of the least common multiple; and since it contains 3, we do not again multiply by 3. Lastly, 5 and 7 must enter into the least common multiple.

The factors of the least common multiple are then 2^4 , 3^2 , 5 and 7; and these, multiplied together, give $2^4 \times 3^2 \times 5 \times 7 = 5040 =$ least common multiple.

Hence, to find the least common multiple of two or more numbers, we have the following:—

RULE.

Resolve the numbers into their prime factors (Art. 21), select all the different factors which occur, observing when the same factor has different powers, to take the highest power. The continued product of the factors thus selected will be the least common multiple.

EXERCISE.

1. What is the least common multiple of 8, 9, 10, 12, 25, 32, 75, and 80?
Here $8 = 2^3$, $9 = 3^2$, $10 = 2 \times 5$, $12 = 2^2 \times 3$, $25 = 5^2$, $32 = 2^5$, $75 = 5^2 \times 3$, $80 = 2^4 \times 5$. Therefore the least common multiple $= 2^5 \times 3^2 \times 5^2 = 7200$.
2. What is the least common multiple of 6, 7, 42, 9, 10 and 630?
Ans. $2 \times 3^2 \times 5 \times 7 = 630$.
3. What is the least common multiple of the nine digits?
Ans. $2^3 \times 3^2 \times 5 \times 7 = 2520$.
4. What is the least common multiple of 6, 9, 12, 15, 18, 21, and 30?
Ans. 1260.
5. What is the least common multiple of 670, 100, 335, and 25?
Ans. 6700.
6. What is the least common multiple of 8, 10, 18, 27, 36, 44, and 396?
Ans. 11880.

SECOND METHOD.

32. We may also find the least common multiple of two or more numbers by the following:—

RULE.

Write the given numbers in a line, with two points between them. Divide by the LEAST number which will divide any two or more of them without a remainder, and set the quotients and the undivided numbers in a line below.

Divide this line and set down the results as before; thus continue the operation till there are no two numbers which can be divided by any number greater than 1.

The continued product of the divisors and the numbers in the last line will be the least common multiple sought.

EXAMPLE 7.—What is the least common multiple of 16, 48, and 108?

$$\begin{array}{r}
 2)16 \cdot 48 \cdot 108 \\
 \hline
 2)8 \cdot 24 \cdot 54 \\
 \hline
 2)4 \cdot 12 \cdot 27 \\
 \hline
 2)2 \cdot 6 \cdot 27 \\
 \hline
 3)1 \cdot 3 \cdot 27 \\
 \hline
 1 \cdot 1 \cdot 9
 \end{array}$$

Ans. $2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 9 = 432 =$ least common multiple.

The least common multiple of 1, 1, and 9 is 9, and the least common multiple of 1, 1, and $9 \times$ by 3, will be the least common multiple of 1, 3, and 27, the numbers of the fifth line; the least common multiple of 1, 3, and 27, \times 2, will be the least common multiple of 2, 6, and 27, the numbers of the fourth line; the least common multiple of 2, 6, and 27, \times 2, will be the least com-

mon multiple of 4, 12, and 27, the numbers in the third line; the least common multiple of 4, 12, and 27, $\times 2$, will be the least common multiple of 8, 24, and 54, the numbers in the second line; and the least common multiple of 8, 24, and 54, $\times 2$ will be the least common multiple of 16, 48, and 108, the given numbers.

The *reason* of the preceding rule depends upon the principle that the least common multiple of two or more numbers, is composed of all the *prime factors* of the given numbers, each taken the *greatest number* of times it is found in either of the given numbers.

NOTE.—In finding the least common multiple by this method, it is necessary to divide by the *smallest* number, which will divide two or more of them without a remainder, because the divisor may otherwise be a composite number (Art. 21), and have a factor *common* to it, and one of the quotients is the last line. Consequently, the continued product of the divisors, and these quotients or undivided numbers in the last line, would be too great for the least common multiple.

Thus, in the third of the following operations the divisor 9 is a composite number, containing the factor 3, common to it and the 3 in the quotient; consequently the product is *three times too large*. In the second operation the divisor 12 is a composite number, and contains the factor 6, common to it and the 6 in the quotient: therefore the product is *six times too large*.

The object of arranging the given numbers in a line, is that all of them may be resolved into their prime factors at the same time; and also to present at a glance the factors that compose the least common multiple required.

EXAMPLE 8.—What is the least common multiple of 12, 18, 36?

| I. | II. | III. |
|--|--------------------------------|--|
| 2)12 .. 18 .. 36 | 12)12 .. 18 .. 36 | 2)12 .. 18 .. 36 |
| 2)6 .. 9 .. 18 | 3)1 .. 18 .. 3 | 2)6 .. 9 .. 18 |
| 3)3 .. 9 .. 9 | 1 .. 6 .. 1 | 9)3 .. 9 .. 9 |
| 3)1 .. 3 .. 3 | 12 \times 3 \times 6 = 216 | 3 .. 1 .. 1 |
| 1 .. 1 .. 1 | | 2 \times 2 \times 9 \times 3 = 108 |
| 2 \times 2 \times 3 \times 3 = 36 = l. c. m. | | |

EXERCISE.

9. Find the least common multiple of 12, 20, and 24. *Ans.* 120.
10. Find the least common multiple of 14, 21, 3, 2, and 63. *Ans.* 126.
11. Find the least common multiple of 18, 12, 39, 216, and 234. *Ans.* 2808.
12. Find the least common multiple of 8, 18, 15, 20, and 70. *Ans.* 2520.
13. Find the least common multiple of 24, 16, 18, and 20. *Ans.* 720.
14. Find the least common multiple of 60, 50, 144, 35, and 18. *Ans.* 25200.
15. Find the least common multiple of 27, 54, 81, 14, and 63. *Ans.* 1134.

THIRD METHOD.

33. The least common multiple of several numbers is most expeditiously found by the following :—

RULE.

Write the given numbers in a line. Take any one of them as divisor, and strike out of each of the given numbers all the factors that are common to it and the assumed number.

Arrange the uncanceled factors of the given numbers, and the uncanceled numbers in a line, take the least other number which exactly contains one or more of them, and strike out all the factors of the numbers in the second line which are common to any of them and the second assumed number.

Proceed thus until the assumed numbers cancel all the factors of the given numbers.

Multiply all the assumed numbers together for the least common multiple of the given numbers.

EXAMPLE 16.—What is the least common multiple of 16, 27, 45, 60, 88, 96, 100.

| | |
|------------|-----------------------------------|
| Assume 100 | 16 · 27 · 45 · 80 · 88 · 96 · 100 |
| Assume 24 | 4 · 27 · 9 · 8 · 22 · 24 |
| Assume 99 | 9 · 2 · 11 |
| | 100 × 24 × 99 = 237600 = l. c. m. |

EXPLANATION.—4, a factor of 100, reduces 16 to 4, 88 to 22, and 96 to 24; 5, another factor of 100, reduces 45 to 9; and 20, another factor of 100, reduces 60 to 3. The numbers in the second line then are 4, 27, 9, 3, 22, and 24. We assume 24, of which a factor, 4, cancels 4; another factor, 2, reduces 22 to 11; and another factor, 3, reduces 27 to 9 and 9 to 3. The numbers in the third line then are 9, 3, and 11. For this line we assumed 99, of which a factor, 3, cancels 3; another factor, 9, cancels 9; and a third, 11, cancels 11.

Now since the least common multiple of a series of numbers is a number which still contains all the prime factors of each number and none others, it is manifest that the least common multiple of the given numbers will be the same as the least common multiple of 100, and 4, 27, 9, 3, 22, and 24, because only those factors which were common to the given numbers and 100 were struck out.

Similarly, the least common multiple of 100, 24, and 9, 3, and 11, will be the same as the least common multiple of 100, and the numbers in the second line, since only those factors which were common to 24 and the numbers of the second line are struck out.

Finally the least common multiple of 100, 24, and 99, is equal to the least common multiple of the given numbers.

EXAMPLE 17.—What is the least common multiple of 120, 40, 39, 65, 88, and 16 ?

| | |
|------------|----------------------------------|
| Assume 120 | 120 · 30 · 39 · 65 · 88 · 16 |
| Assume 13 | 13 · 12 · 11 · 2 |
| Assume 22 | 11 · 2 |
| | 120 × 13 × 22 = 34320 = l. c. m. |

EXPLANATION.—We first assume 120. Now this cancels 120 and 40. Also, 3, a factor of 120, reduces 39 to 13, and 5, another factor, reduces 65 to 13. Also 8, another factor, reduces 88 to 11 and 16 to 2. Next assume 13, this cancels 13 and 13. Next assume 22, of which 11, one factor, cancels the 11, and another factor 2, cancels 2.

EXAMPLE 18.—Find the least common multiple of 12, 16, 20, 24, 30, 48, 56, and 64.

$$\begin{array}{l} \text{Assume } 96 \left\{ \begin{array}{l} 12 \cdot 16 \cdot 20 \cdot 24 \cdot 30 \cdot 48 \cdot 56 \cdot 64 \\ \end{array} \right. \\ \text{Assume } 70 \left\{ \begin{array}{l} \\ \end{array} \right. \\ \hline 96 \times 70 = 6720 = 1. \text{ c. m.} \end{array}$$

EXERCISE.

19. What is the least common multiple of 300, 200, 150, 50, 60, 75, and 125? *Ans.* 3000.
 20. What is the least common multiple of 20, 60, 15, 165, 210, 63, and 27? *Ans.* 41580.
 21. What is the least common multiple of 12, 132, 144, 60, 96, and 1728? *Ans.* 95040.

Work also by this method all the preceding questions in least common multiple.

DIFFERENT SCALES OF NOTATION.

34. The *radix* or *base* of a scale of notation is its common ratio. Thus, in our system the *radix* is 10; in the duodecimal system the *radix* is 12, &c.

35. If the expression 12345 represents a number in the common or decimal scale of notation, we read it twelve thousand three hundred and forty-five; but if it expresses a number in any other scale, we cannot so read it, because the names *thousands*, *hundreds*, &c., belong only to the decimal scale. In order to read it properly in any other scale we should have to *invent* names for the different orders. In place, however, of doing this, we simply read over the digits and indicate the scale. For example, if the expression 24678 be a number in the *nonary* scale, we read it thus—*two, four, six, seven, eight* in the *nonary* scale.

36. We may express the number 4578 (decimal scale) by writing the order of each digit beneath it, thus,

$$\begin{array}{cccc} 4 & 5 & 7 & 8 \\ 10 & 10 & 10 & 10 \\ \hline & 3 & 2 & 1 \end{array}$$

and then read it 8 units, 7 of the order of tens, 5 of the order of hundreds or tens squared, or second order of tens 4 of the third order of tens, &c. Similarly if 4578 express a number in the *nonary* scale, we may write it

$$\begin{array}{cccc} 4 & 5 & 7 & 8 \\ 9 & 9 & 9 & 9 \\ \hline & 3 & 2 & 1 \end{array}$$

and read it 8 units, 7 nines, 5 of the *second order* of nines, 4 of the *third order* of nines, &c.

37. The expression 10 always represents the radix of the scale. In the *decimal* scale 10 is equal ten; in the *binary* scale 10 is equal two; in the undenary scale 10 is equal eleven, &c.

38. It is obvious that, in any scale, the highest digit used must be *one less* than the radix. Thus, in the *decimal* scale, the highest digit is 9; in the *ternary*, 2; in the *octenary*, 7, &c. In writing numbers in the duodenary scale we use the letter *t* to represent *ten*, and *e*, *eleven*.

39. Let it be required to reduce 337 from the *decimal* to the *octenary* scale.

OPERATION. EXPLANATION.—If we divide 337 by 8, we distribute it into 42 groups of 8 each, and have a remainder of 1 unit. If now we divide these groups of 8 by 8, we obtain 5 groups of a still higher order, each containing 8 of the former groups, with a remainder of 2 of these groups.
 8)337
 42—1
 5—2
 337, in the decimal scale, is therefore equal to 501 in the octenary scale; i.e., the successive remainders written in order constitute the equivalent expression in the required scale.

Hence, to reduce a number from one scale to another, we have the following:—

RULE.

Divide the number continually by the radix of the proposed scale, till the quotient is less than the radix.

Write all the remainders, thus obtained, in regular order from left to right, beginning with the last, and placing 0s where there are no remainders. The result will be the required number.

EXAMPLE 1.—Reduce 7342 from the common to the *quinary* scale.

OPERATION.

5)7342
 1468—2
 293—3
 58—3
 11—3
 2—1

Therefore 7342 *denary* = 213332 *quinary*.

EXAMPLE 2.—Express nine millions, three hundred and forty-two thousand and twenty-seven, in the *duodenary* scale.

OPERATION.

12)9342027

12)778502—3

12)64875—2

12)5406—3

12)450—6

12)37—6

3—1

Therefore 9342027 *denary* = 3106323 *duodenary*.

EXERCISE.

3. Change 592835 from the decimal to the *duodenary* scale. *Ans.* 2470te.
4. Express the common number 3700 in the *quinary* scale. *Ans.* 104300.
5. Express 10000 in the *undenary* scale. *Ans.* 7571.
6. Express a million in the *senary* scale. *Ans.* 33233344.
7. Express 10000 in the *octenary* scale. *Ans.* 23420.
8. Transform 12345654321 into the *duodenary* scale. *Ans.* 248664et69.
9. Express 10000 in the *nonary* scale. *Ans.* 14641.
10. Transform 300 from the common to the *binary* scale. *Ans.* 100101100.

EXAMPLE 11.—Transform 2313042 from the *quinary* to the *octenary* scale.

OPERATION.

8)2313042

8)131310—7

8)10100—5

8)311—2

8)20—1

1—2

EXPLANATION.—We divide here as before, bearing in mind, however, that the ratio is no longer ten, but *five*. We proceed thus.—8 in 2, no times; twice five (the radix) is ten, and 3 make thirteen; 8 in 13, 1 and 5 over; 5 times 5 are 25, and 1 make 26; 8 in 26, 3 times and 2 over; twice 5 are 10, and 3 make 13, 8 in 13, once and 5 over, &c.

Therefore 2313042 *quinary* = 121257 *octenary*.

NOTE.—The Roman Numeral written over the number indicates the radix of the scale.

EXAMPLE 12.—Transform 378t13 from the *undenary* to the *duodenary* scale.

OPERATION.

$$\begin{array}{r} \text{XI.} \\ 12)378t13 \\ \hline \end{array}$$

$$12)34456-8$$

$$12)3132-4$$

$$12)294-9$$

$$12)26-9$$

$$12)2-4$$

Observe, the first two figures here are not thirty-seven, but $3 \times 11 + 7 = 40$. We say 12 into 40, 3 times and 4 over; next, 12 into $4 \times 11 + 8$ or 52, &c.

378t13 *undenary* = *duodenary*, 249943. *Ans.*

EXAMPLE 13.—Transform t423t from the *duodenary* to the *nonary* scale.

OPERATION.

$$\begin{array}{r} \text{XII.} \\ 9)t423t \\ \hline \end{array}$$

$$9)11971-1$$

$$9)1649-4$$

$$9)206-3$$

$$9)28-6$$

$$3-5$$

Observe, here we say 9 into t, ten, 1 and 1 over; 9 into 16, ($1 \times 12 + 4$) 1 and 7 over; 9 into 86, ($7 \times 12 + 2$) 9 and 5 over; 9 into 63, ($5 \times 12 + 3$) 7; 9 into t, 1 and 1 over.

And we proceed in the other lines in the same manner.

t423t *duodenary* = 356341 *nonary*.

EXERCISE.

14. Transform 37704 from the *nonary* to the *octenary* scale.
Ans. 61415.
15. Transform 444 and 4321 from the *quinary* to the *septenary* scale.
Ans. 235 and 1465.
16. Transform 1212201 from the *quaternary* to the *nonary* scale.
Ans. 10000.

40. A number may be transformed from any scale to the decimal by the preceding rule, but the following is more convenient.

Multiply the left hand figure by the given radix, and to the product add the next figure.

Then multiply this sum by the radix and add the next figure. Continue this process until all the figures have been used. Then the last product will be the number in the decimal scale.

NOTE.—Both this and the preceding rule are the same in principle as reducing denominate numbers from one denomination to another.

EXAMPLE 17.—Reduce 76345 from the *octenary* scale to the *decimal* scale.

OPERATION.
 VIII.
 76345
 8
 —
 62 of the fourth order.
 8
 —
 499 of the third order.
 8
 —
 3996 of the second order.
 8
 —
 31973 units = required number in *decimal* scale.

EXAMPLE 18.—Transform *ettete* from the *duodenary* to the common or *decimal* scale.

OPERATION.
 XII.
 ettete
 12
 —
 142 = number of fifth order.
 12
 —
 1714 = number of fourth order.
 12
 —
 20579 = number of third order.
 12
 —
 246958 = number of second order.
 12
 —
 2963507 = units = required number in *decimal* scale.

EXERCISE.

19. Change 20212331 from the *quaternary* into the *decimal* scale.
Ans. 35261.
20. Change 101202220 from the *ternary* into the *decimal* scale.
Ans. 7854.
21. Transform 1522365 from the *nonary* into the *decimal* scale.
Ans. 841568.
22. Transform 33233344 from the *senary* into the *decimal* scale.
Ans. 1000000.

EXAMPLE 23.—Transform 2734, *octenary* scale, into the *undenary*, *septenary*, and *quinary* scales, and prove the results by reducing all four numbers to the *decimal* scale.

| VIII. | VIII. | VIII. |
|----------|---------|---------|
| 11)2734 | 7)2734 | 5)2734 |
| 11)210—4 | 7)326—2 | 5)454—0 |
| 11)14—4 | 7)36—4 | 5)74—0 |
| 1—1 | 4—2 | 5)14—0 |

| 2734 | 1144 | 1242 | 25000 |
|------|------|------|-------|
| 8 | 11 | 7 | 5 |
| 23 | 12 | 30 | 12 |
| 8 | 11 | 7 | 5 |
| 187 | 136 | 214 | 60 |
| 8 | 11 | 7 | 5 |

2—2
Therefore *octenary* = *undenary* = *septenary* = *quinary*.

1500 *denary*. 1500 *denary*. 1500 *denary*, 1500 *denary*.

Since the results all agree when reduced to the *denary* scale, we conclude the work is correct.

24. Transform 132713 *nonary*, into the *ternary*, *duodenary*, and *octenary* scales, and prove the results by reducing all four numbers to the *denary* scale.

25. Transform 121290 *duodenary*, into the *nonary*, *senary*, *quaternary*, and *binary* scales, and prove the result by reducing all five numbers to the *decimal* scale.

FUNDAMENTAL RULES.

41. The fundamental rules of arithmetic are carried on in the different scales as with numbers in the ordinary or decimal scale; observing that, when we wish to find what to *carry* in addition, subtraction, multiplication, &c., we divide, not by *ten*, but by the radix of the particular scale used.

EXAMPLE 26.—Add together 34120, 3121, 13102, 31410, 12314, 112243 and 444444 in the *senary* scale.

| | |
|------------|---|
| OPERATION. | Observe, the sum of the first line is 14, which, divided by 6 the radix of the scale, gives us 2 to set down and 2 to carry; the sum of the second line is 16, which, divided by the radix, 6, gives us 4 to set down and 2 to carry, &c. |
| VI. | |
| 34120 | |
| 3121 | |
| 13102 | |
| 31410 | |
| 12314 | |
| 112243 | |
| 444444 | |
| 1144042 | <i>Ans.</i> |

EXAMPLE 27.—From 43176 take 9109, in the *undenary* scale.

OPERATION. Observe, here we say 9 from 6, we cannot, but 9 from 17 (I borrowed = 11 and 6) and 8 remains, &c.

| |
|-------|
| XI. |
| 43176 |
| 9109 |
| 35068 |

EXAMPLE 28.—Multiply 342_6 by 567 , in the *octenary* scale.

OPERATION.

$$\begin{array}{r} \text{VIII.} \\ 342_6 \\ \underline{567} \\ 30632 \\ 25204 \\ \underline{21556} \end{array}$$

Observe, we say 7 times 6 are 42, 8 (the radix) into 42 5 to carry and 2 to set down; 7 times 2 are 14 and 5 make 19, equal to 3 to set down and 2 to carry, &c.

2460472 Ans.

EXAMPLE 29.—Divide 671384 by 7876 , in the *nonary* scale.

OPERATION.

$$\begin{array}{r} \text{IX. IX.} \\ 7876 \overline{)671384} (757501 \text{ Ans.} \\ \underline{61786} \\ 52424 \\ \underline{43823} \\ 7501 \end{array}$$

Here 7876 will go into 671384 7 times, (observe it would go 8 times in the *decimal* scale); and 7876 multiplied by 7 gives 61786 , this being subtracted, gives a remainder, 5242 , to which we bring down the next digit, 4 , and proceed as in common division.

NOTE.—After the units' figure is brought down, we may either write the remainder in the form of a fraction, as in example 29, or we may place a point, and annexing 0s, continue the division as in the following example.

Observe, this point is called the *decimal* or *denary point* only in the decimal system. In every other scale of notation it takes its name from the system—thus, in the duodenary or duodecimal system it is called the *duodenary* or *duodecimal point*, in the senary system, the *senary point*, &c.

EXAMPLE 30.—Divide $t134567$ by $e473$, in the *duodenary* scale.

$$\begin{array}{r} \text{OPERATION.} \\ e473 \overline{)t134567} (t7t.e, \&c. \\ \underline{95t06} \\ 753e6 \\ \underline{67829} \\ 97897 \\ \underline{95t06} \\ t791'0 \\ e47'3 \\ \underline{e45'90} \\ t52'79 \end{array}$$

EXERCISE.

31. Multiply 252 by 252 , in the *senary* scale. Ans. 122024 .
32. Divide $32e75721$ by $62te$, in the *duodenary* scale. Ans. $62te$.
33. From 201210 take 102221 , in the *ternary* scale. Ans. 21212 .
34. Multiply 57264 by 675 , in the *octenary* scale. Ans. 51117344
35. Add together 101 , 1001 , 1111 , 1011 , 1000 , 1111 , and 10101 , in the *binary* scale. Ans. 1010100 .

36. Divide 142613 by 2143, in the septenary scale.
Ans. 50·5254+.
37. Add together 65432, 43210, 1444, 65001, and 54321, in the septenary scale.
Ans. 326041.
38. From 7t348 take 5e6t4, in the duodenary scale. *Ans.* 1t864.
39. Multiply 34t7 by 6666, in the duodenary scale
Ans. 1t36e296.
40. Divide 1010100001 by 100101, in the binary scale.
Ans. 10010₁₀₀₁₀₁.

42. All the methods of proof given in Sec. II., for the fundamental rules in the common scale, apply to the various other scales; but it must be remembered that, in using the principle of the proof by *nines* for multiplication and division, we use, not *nine*, but a number one less than the radix of the scale.

Thus, in applying this principle to the proof in Example 34, *sevens* cast out of 57264, give a remainder 3; *sevens* cast out of 675, give a remainder 4. 4×3 , and *sevens* cast out, give a remainder 5; *sevens* cast out of 51117344, give a remainder 5.

If the radix be 12, we cast out the 11s; if the radix be 6, we cast out the 5s, &c.

43. Numbers containing digits to the right of the separating point, are dealt with according to the rules given in Arts. 53 and 88, Sec. II.

EXAMPLE 41.—Multiply 37·14t3 by 6·1et in the duodenary scale.
 OPERATION. We place the separating point in the product so as to have 37·14t3 seven digits to the right of it, because there are four to the right of the point in the multiplicand and three in the multiplier, and $4+3=7$. (Art. 53, Sec. II.)

2ee2066
 3363549
 3714t3
 1968516

1t1·t08e836

DUODECIMAL MULTIPLICATION.

44. The term *duodecimal* is commonly applied to a set of denominate fractions having 1 foot (*linear, square, or cubic* measure) for their unit.

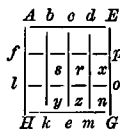
The foot is supposed to be divided into 12 equal parts, called *primes*; each of which is divided into 12 equal parts, called *seconds*, &c.

TABLE.

| | | | | | |
|---------------|------|---|---------|--------|-----|
| 12 fourths''' | make | 1 | third, | marked | ''' |
| 12 thirds | " | 1 | second, | " | " |
| 12 seconds | " | 1 | prime, | " | ' |
| 12 primes | " | 1 | foot | " | ft. |

45. The term "inch," sometimes used in this table, is objectionable, corresponding to "prime" only when the unit is a linear foot. When the unit is a square foot, the prime is $\frac{1}{12}$ of a square foot, or is a *surface* 12 inches long and 1 inch wide; when the unit is a cubic foot, the prime is $\frac{1}{12}$ of a cubic foot, or is a *solid* 12 inches long, 12 inches wide, and 1 inch thick.

46. Let $A E H G$ represent the surface of a rectangular table four feet in length and three in breadth. Now, if $A E$ be divided into four equal parts, and $A H$ into three equal parts, each of these parts, $A b, b c, c d, d E$, will be 1 foot long, and if lines $b k, c e, d m$ are drawn through b, c , and d , parallel to $A H$, and lines $f p, l o$ through l and o , parallel to $A E$, they will divide the whole surface into the small figures, $A b s f, b s r c, \&c.$



And, since $A b = 1$ foot, and $A f = 1$ foot, $A f s b$ is a square foot, so likewise is each of the other figures, $b s r c, c r x d, \&c.$

Now it is evident that there are as many vertical rows of these square feet as there are linear feet in $A E$, and as many squares in each row as there are linear feet in $A H$, that is in this case the number of square feet in the surface $= 4 \times 3 = 12$.

As the same method of proof would apply in any similar case, it appears that—

The area of any rectangular surface is found in square feet, and fractions of a square foot, by multiplying the number expressing how many linear feet, &c., there are in the length, by the number expressing how many linear feet, &c., there are in the breadth.

NOTE.—In linear measure, primes are linear inches; in square measure, seconds are square inches; and in cubic measure, thirds are cubic inches.

47. Problem 41, page 143, is, in effect, equivalent to finding the area of a rectangle, one side of which is 43 feet 1' 4" 10''' and 3'''' long, and the other 6 ft. 1' 11" 10''' long. The answer may be translated 265 sq. ft. 0' 8" 11'''' 8'''' 3'''''' and 6''''''.

NOTE.—1 $\frac{1}{2}$, the number to the left of the separating point, is a number in the duodenary scale. In order to read it in common terms, we convert it to an equivalent number in the decimal scale (Art. 39), and thus obtain 265. It is obvious that, since the orders primes, seconds, thirds, &c., form a series of numbers descending in a 12-fold proportion from left to right, we must allow the digits to the right of the point to remain as they are.

EXAMPLE 42.—Find the area of a rectangular ceiling 43 ft. 4" 7''' long by 20 ft. 11' 10''' wide.

OPERATION. Here, since 43 and 20 are numbers in the common scale, we must reduce them to the duodenary scale before attaching them by the point to the other parts of the numbers. We thus obtain, for the first, 37, and for the second, 18. After multiplying and pointing off four places in the product, we find 63 $\frac{1}{2}$ to the right of the point; this, reduced to an equivalent number in the common scale, gives us 910, to which we attach the other four digits, with their indices, as below.

$$63\frac{1}{2}502\frac{1}{2} = 910 \text{ sq. ft. } 5' 0'' 2''' 10'''' \text{ Ans.}$$

48. The common arithmetical rule for duodecimal multiplication is as follows:—

RULE.

Write the multiplier under the multiplicand having quantities of the same denomination under each other.

Multiply each term of the multiplicand by each term of the multiplier separately.

Write the partial products under one another, so as to have quantities of the same name in the same vertical column, and add the several partial products together.

NOTE.—Considering the foot to have no index, the denomination of the product of any two factors is found by adding their indices.

Thus, $3'' \times 2'''$ give $6''''$; $4 \text{ ft.} \times 7''''$ give $28''''$; $2 \text{ ft.} \times 3 \text{ ft.}$ give 6 ft. ; $9' \times 11'$ give $99'$, &c.

This is commonly expressed, for the sake of brevity, by saying—feet into feet produce feet, feet into primes produce primes, &c., primes into feet produce primes, primes into primes produce seconds, &c., seconds into seconds produce fourths, seconds into thirds produce fifths, &c.

EXAMPLE 43.—Multiply $43 \text{ ft. } 4' 7''$ by $20 \text{ ft. } 11' 10''$.

| | |
|----------------------------|--|
| OPERATION. | |
| $43 \ 4' \ 7''$ | |
| $20 \ 11 \ 10$ | |
| $3 \ 0 \ 1 \ 5'' \ 10''''$ | |
| $39 \ 9 \ 2 \ 5$ | |
| $867 \ 7 \ 8$ | |

Here 7 and 10, multiplied together, give us 70, and adding their indices, we see that the product is so many fourths— $70''''$, are equal to $10''''$ to set down and $5''''$ to carry. Next $4' \times 10'' = 40''$ and $5''$ make $45'' = 3'' 9''$, &c.

$910 \ 5' \ 0'' \ 2'' \ 10''''$

49. In comparing this example with 42 it will be seen that the two methods very closely agree—the only difference being that, in the latter method, upon reaching the units or feet, we drop the duodecimal scale and carry on the process in the decimal scale, while, in the former, we carry on the whole process in the duodecimal scale, and afterwards reduce that part of the expression to the left of the separating point to the common or decimal scale.

50. Provided we multiply every part of the multiplicand by every part of the multiplier, it is perfectly immaterial where we commence the process. It is customary, however, to commence, not as we have done in the last example, with the lowest denomination of both multiplier and multiplicand, but with the highest of the multiplier and the lowest of the multiplicand. Hence duodecimal multiplication is frequently called *Cross Multiplication*.

EXAMPLE 44.—Multiply 3 ft. 2' 7" 4''' by 1' 3" 7'''

| OPERATION. | | | | | | |
|--------------------------------------|----|----|-------|---------|--|--|
| 3 ft. | 2' | 7" | 4''' | | | |
| 1 | 3 | 7 | | | | |
| 3 | 2 | 7 | 4''' | | | |
| 9 | 7 | 10 | 0'''' | | | |
| 1 | 10 | 6 | 3 | 4'''''' | | |
| 4' 2" 1''' 8'''' 3''''' 4'''''' Ans. | | | | | | |

EXERCISE.

45. Multiply 4 ft. 7' 6" 10''' by 9 ft. 7' 11" 11'''.
Ans. 44 sq. ft. 9' 1" 8''' 0'''' 5'''''' 2''''''.
46. Multiply 19 ft. 10' 3" by 11 ft. 2' 7".
Ans. 222 sq. ft. 8' 0" 5''' 9''''.
47. Multiply 9' 7''' 4'''' by 7''' 3'''' 11''''.
Ans. 5'''' 10'''' 4'''''' 11'''''' 8'''''' 8''''''.
48. How many square inches, &c., are there in a sheet of paper 9½ inches and 5 inches 7" 4''' wide?
Ans. 4' 6" 8''' 6'''' or 54½ sq. inches.
49. What is the superficial contents of a sheet of glass whose length is 7 ft. 4' 11" and breadth 3 ft. 2' 2" ?
Ans. 23 sq. ft. 6' 9" 7''' 10''''.

51. The solid contents are found by multiplying together the length, breadth, and thickness.

EXAMPLE 50.—How many cords of wood are there in a pile 79 ft. 8 inches long, 4 ft. 2 inches wide, and 7 ft. 11 inches high?

| FIRST METHOD. | SECOND METHOD. |
|---------------|-----------------|
| 678 | 79 8' |
| 42 | 4 2' |
| 1134 | 13 3' 4'' |
| 2268 | 318 8' |
| 237e4 | 331 11' 4'' |
| 7e | 7 11' |
| 214348 | 304 3' 4'' 8''' |
| 141774 | 2323 7' 4'' |

No. of ft. in cord = 18)162e f88(18'64469 *duodenary*. 2627 11' 8" 8''' = 128
 78 = 20½ ¾ ¼ ½ com. scale. (number of ft. in cord)
 = 20½ ¾ ¼ ½ cords. *Ans.*

76e
 714
 57f
 540
 378
 3'68 &c.

* 1½ + 1¼ + 1⅓, &c. of a square foot.

EXERCISE.

51. Multiply together 15 ft., 1 ft. 2', and 8'.
Ans. 11 cubic ft. 8' = 11 cubic ft, 1152 cubic in.
52. Multiply together 53 ft. 6 in., 10 ft. 3 in., and 2 ft.
Ans. 1096 cubic ft. 9'.
53. How many cords of wood in a pile 10 ft. long, 5 ft. high, and 7 ft. wide?
Ans. 2 cords 94 cubic ft.
54. How many cords of wood are there in a pile 4 ft. wide, 5 ft. 3 in. high, and 70 ft. long?
Ans. $11\frac{3}{4}$.
55. What are the exact cubic contents of a block of marble 4 ft. 7' 8" long by 9 ft. 6' wide and 2 ft. 11' thick?
Ans. 128 cubic ft. 6' 5" 2".
56. How many bricks, 8 inches long, 4 inches wide, and 2 inches thick, will it require to make a wall 25 ft. long, 20 ft. high, and 2 ft. 6 inches thick?
Ans. 33750 bricks.

52. It is sometimes asked how we can multiply feet, inches, &c., by feet, inches, &c., while we cannot multiply pounds, shillings and pence by pounds, shillings and pence. The answer is very simple.

1st. When we say that feet multiplied by feet give square feet, we merely use, as we have seen, (Art. 46), an abbreviated form of expression for the following, viz: that "the number of square feet contained in any rectangular surface, is equal to the product of two numbers, one of which represents the number of linear feet in one side; and the other, the number of linear feet in the adjacent side."

2nd. When we are multiplying together primes, seconds, &c., we are merely multiplying together a set of factors having 12 or powers of 12 for denominators; and when we say that *seconds* multiplied by *fourths*, give *sixths*; *primes*, multiplied by *seconds*, give *thirds*, &c., we simply mean that the product of any two of these fractions is a fraction having for its denominator a power of 12, which power is indicated by the sum of the indices of the factors.

It is hence obvious that duodecimal multiplication affords no support whatever to the idea that money may be multiplied by money.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers after the questions refer to the articles of the Section.

1. What is a *measure* of a number? (1)
2. What is a *multiple* of a number? (2)
3. What is an *integer*? (3)
4. Of how many kinds are integers? (4)
5. What is an *even* number? (5)
6. What is an *odd* number? (6)
7. What is a *prime* number? (7)
8. What is a *composite* number? (8)
9. What are the *factors* of a number? (9)
10. By what other names are factors known? (10)
11. What is a *common measure* of two or more numbers? (11)
12. When are two or more numbers *prime to each other*? (12)
13. Are all prime numbers prime to each other? (12)
14. Are all composite numbers prime to each other? (12)
15. What are *commensurable* numbers? (13)
16. What are *incommensurable* numbers? (14)
17. What is a *square* number? (15)

18. What is a *cube* number? (16)
19. What is a *perfect* number? (17)
20. Mention some perfect numbers. How do all perfect numbers terminate? (17)
21. What are *amicable* numbers? Mention some amicable numbers.
22. What is meant by the *properties of numbers*? (19)
23. What is the sum of two or more even numbers? (19-I.)
24. What is the difference of two even numbers? (19-II.)
25. What is the sum of 3, 5, 7, &c., odd numbers? (19-IV.)
26. What is the sum of 2, 4, 6, 8, &c., odd numbers? (19-V.)
27. What is the sum or difference of an odd and an even number? (19-VI.)
28. When is the product of any number of factors even? (19-IX.)
29. When is the product of any number of factors odd? (19-XI.)
30. When will a number measure the *sum, difference* and product of two numbers? (19-XIII.)
31. If the number 9 be multiplied by any single digit to what is the sum of the digits in the product equal? (19-XVI.)
32. By what is any number ending in 0 divisible? (19-XIX, &c.)
33. By what is any number ending in 5 divisible? (19-XX.)
34. By what is any number ending in 2 divisible? (19-XIX.)
35. When is a number divisible by 4? (19-XXII.)
36. When is a number divisible by 8? (19-XXIII.)
37. When is a number divisible by 9? (19-XXIV.)
38. When is a number divisible by 3? (19-XXV.)
39. When is a number divisible by 11? (19-XXVI.)
40. Show that every composite number may be resolved into prime factors. (19-XXVII.)
41. Show that the *least* divisor of any number is a prime number. (19-XXVIII.)
42. With what digits must all prime numbers except 2 and 5 terminate? (19-XXXI.)
43. How do you find the prime numbers between any limits? (20)
44. What is this process called and why? (20)
45. When it is required to ascertain whether a given number is prime or not, what is the first thing we do? (20)
46. When we try the primes of the table as divisors, which is the highest we need use? (20)
47. Why is it unnecessary to try any divisor greater than the square root of the number? (20)
48. How do we resolve a composite number into its prime factors? (21)
49. By what numbers can a composite number be divided? (21-Note.)
50. What is the rule for finding all the divisors of a number? (22)
51. How do we find simply *how many* divisors a number has? (23)
52. What is the greatest common measure of two or more numbers? (24)
53. How do we find a common measure of two or more numbers? (25)
54. How do we find the greatest common measure of two numbers? (26)
55. Prove the rule in Art. 26.
56. How do we find the G. C. M. of three or more numbers? (27)
57. What is the second method of finding the G. C. M.? (28)
58. Upon what principle does this method rest? (28)
59. What is a common multiple of two or more numbers? (29)
60. What is the least common multiple of two or more numbers? (30)
61. Give the first rule for finding the l. c. m. of two or more numbers. (31)
62. Give the second rule. (32) What is the reason of this rule? (32)
63. Give the most convenient and expeditious rule for finding the l. c. m. of several numbers. (33)
64. What is meant by the *radix* or *base* of a system of notation? (34)
65. How do we read numbers in different scales? (35)
66. Express the number 234213 quinary as in Art. 36.
67. What does the expression 10 always represent? (37)
68. What is the highest digit used in any scale? (38)
69. How do we reduce a number from one scale to another? (39)

70. What is the rule for transforming a number from any scale into the decimal? (40)
71. How are the fundamental operations carried on in the different scales? (41)
72. How is the separating point named in the different scales? (41-Note.)
73. How are operations in the different scales proved? (42)
74. What are duodecimals? (44)
75. Give the table of duodecimals. (44)
76. What is a prime? (45)
77. How is the area of a rectangular surface found? (46)
78. What is the rule for duodecimal multiplication? (48)
79. How may the rule for finding the denomination of the product be concisely worded? (48)
80. How are solid contents found? (51)
81. Show that duodecimal multiplication affords no support to the idea that money may be multiplied by money, &c. (52)

MISCELLANEOUS EXERCISE.

(On preceding Rules.)

1. Add together \$729.18, \$710.50, \$166.78, £93 14s. 7½d. £276 19s. 10½d., \$497.81 and £275 4s. 11¾d.
2. Multiply 47 miles, 6 fur. 17 per. 4 yds. 2 ft. 7 in. by 576.
3. How many divisors has the number 243000?
4. From 713427 *octenary* take 4234434 *quinary* and give the answer in both scales.
5. Divide 79.342 by .00006378.
6. Express 79423 and 234567 in Roman numerals.
7. What is the l. c. m. of 5, 7, 9, 11, 15, 18, 20, 21, 22, 24, 28, 30, 33, 35, 36, 40, 42, 44, 45, 48, and 50.
8. Give all the readings of 376.342.
9. Multiply 64276.3427 by 9999993000.
10. Transform 78263 *nonary* into the *quinary* and *undenary* scales and prove the results by reducing all the numbers to the *septenary* scale.
11. Form a table of all the prime numbers less than 200.
12. Reduce £672 7s. 7d. to dollars and cents.
13. What is the G. C. M. of 243000, 891, 37800 and 35100.
14. Give all the readings of 6 yards 3 qrs. 3 nails 2 inches.
15. Write down as one number, seven hundred and forty-two quintillions, nine hundred and five billions, seventy-eight thousand and fourteen, and eighty-seven million, two hundred thousand and eleven tenths of trillions.
16. Read the following numbers:
71300100200401.000000070402
134900101000100100.000200020002
4700000000020007.00000000000278.
17. Add together £178 16s. 4¾d., £97 15s. 11½d., £693 19s. 11½d., £216 11s. 9¼d., £678 14s. 7½d., £197 13s. 11¼d., £117 6s. 5d., and £91 1s. 1¾d.
18. What are the prime factors of 276000.

19. Multiply 6 ft. 2' 7" 9''' 10'''' by 13 ft. 11' 11" 11''' 7''''.
20. Divide $7te9\cdot047$ by $713t96$ in the *duodenary* scale.
21. What number in the common scale is the greatest that can be expressed by seven figures in the *quaternary* scale?
22. What number in the common scale is the least that can be expressed as an integral number by five figures in the *octenary* scale?
23. Reduce 74002702 square inches to acres.
24. What is the least common multiple of 240, 780, 1620 and 1728?
25. Divide \$7894.16 among 3 men, 4 women and 6 children, so that each woman shall have twice as much as a child and each man 5 times as much as a woman. What is the share of each?
26. What are the greatest and least integral numbers in the common scale that can be expressed by 10 figures in the *binary* scale?
27. Divide 729 yds. 3 qrs. 3 na. 1 in. by 7 yds. 1 qr. 1 na. 1 in.
28. Multiply $702\cdot4978$ by $63\cdot423$.
29. From 723426 take $938\cdot9126141$.
30. From 129 lb. take 63 lb. 4 oz. 7 drs. 2 scr.
31. What are the divisors of 1064?
32. How many yards of carpet 2 ft. 7 in. wide, will be required to cover a floor 30 ft. 6 in. long and 20 ft. 11 in. wide?

SECTION IV.

VULGAR AND DECIMAL FRACTIONS, &c.

1. A fraction is an expression representing one or more of the equal parts into which any quantity may be divided

2. If a quantity be divided into 2, 5, 9 or 34, &c equal parts, then *one* of these parts is called *one-half*, *one-fifth*, *one-ninth*, or *one-thirty-fourth*, &c., as the case may be.

| | |
|--|--|
| one-half is written..... $\frac{1}{2}$ | one-ninth is written..... |
| one-third is written..... $\frac{1}{3}$ | one-hundredth is written $\frac{1}{100}$ |
| one-fourth is written..... $\frac{1}{4}$ | one-sixty-eighth is written $\frac{1}{68}$ |
| one-fifth is written..... $\frac{1}{5}$ | eleven-seventenths is written $\frac{11}{7}$, &c. |

3. The division of one number by another may be i

icated in three different ways, viz: by using the full sign of division, \div , or either of its parts, —, or :

Thus we may indicate the division of 17 by 8, by writing them thus $17 \div 8$, or thus $17 : 8$, or thus $\frac{17}{8}$.

Now the last of these, viz: $\frac{17}{8}$ is a fraction, and so in every other case, a fraction indicates the division of one number, called the *numerator*, by another number, called the *denominator*.

4. In a fraction the number below the line is called the denominator, because it indicates into how many equal parts the unit is divided,—i. e., it tells the *denomination* of the parts. The number above the line is called the numerator, because it *num-rates* or tells how many of these equal parts are to be taken. (Art. 2)

5. The numerator and denominator are called the *terms* of the fraction.

6. Since every fraction expresses the division of the numerator by the denominator, it follows that—

The *value* of the fraction is the *quotient* obtained by dividing the numerator by the denominator.

7. Hence, 1st. When the numerator is less than the denominator, the value of the fraction is less than 1.

2nd. When the numerator is equal to the denominator the value of the fraction is equal to 1.

3rd. When the numerator is greater than the denominator the value of the fraction is greater than 1.

8. From (Art. 6) and (Arts. 79-84, Sec. II.) it is manifest that—

1st. Multiplying the numerator of a fraction by any number multiplies the fraction by that number.

2nd. Multiplying the denominator of a fraction by any number divides the fraction by that number.

3rd. Multiplying both numerator and denominator of a fraction by the same number does not affect the value of the fraction.

4th. Dividing the numerator of a fraction by any number divides the fraction by that number.

5th. Dividing the denominator of a fraction by any number multiplies the fraction by that number.

6th. Dividing both numerator and denominator of a fraction by the same number does not affect its value.

9. Fractions are divided into two classes:—vulgar and decimal.

10. A Decimal Fraction is a fraction in which the denominator is 1, followed by one or more 0s.

11. All other fractions are called Vulgar or Common Fractions.

NOTE.—The word vulgar is here used in the sense of common.

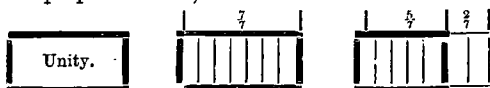
12. There are six kinds of vulgar fractions—*proper*, *improper*, *mixed*, *simple*, *compound*, and *complex*.

13. A Proper Fraction is one in which the denominator is greater than the numerator.

A Proper Fraction may also be defined to be a fraction whose value is less than 1.

Thus $\frac{1}{3}$, $\frac{4}{8}$, $\frac{7}{8}$, $\frac{12}{16}$, $\frac{13}{14}$, $\frac{9}{10}$ are proper fractions.

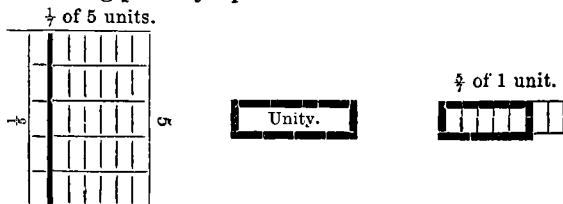
The following diagrams represent *unity*, *seven-sevenths*, and the *proper fraction*, *five-sevenths*.



The very faint lines indicate what $\frac{5}{7}$ wants to make it *equal* to unity, and *identical with* $\frac{7}{7}$. In the diagrams which are to follow, we shall, in this manner, generally subjoin the difference between the fraction and unity.

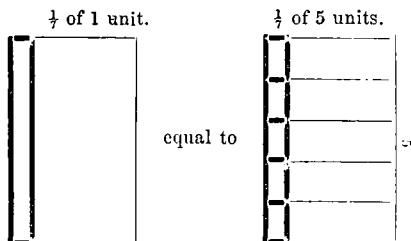
The teacher should impress on the mind of the pupil that he might have chosen any *other* unity to exemplify the nature of a fraction.

14. The following will show that $\frac{5}{7}$ may be considered as either the $\frac{5}{7}$ of 1 or the $\frac{1}{7}$ of 5, both—though not identical—being perfectly equal.



In one case we may suppose that the five parts belong to but 1 unit; in the other, that each of the five belongs to different units of the same kind.

Lastly, $\frac{5}{4}$ may be supposed as the $\frac{1}{4}$ of one unit five times as large as the former; thus—



15. An Improper Fraction is a fraction whose denominator is *not* greater than its numerator.

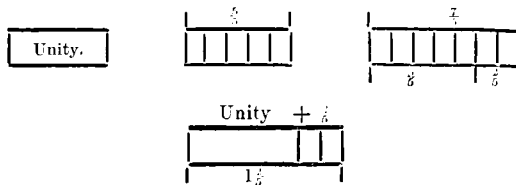
An Improper Fraction may also be defined to be a fraction whose value is equal to or greater than 1.

Thus, $\frac{3}{2}$, $\frac{16}{7}$, $\frac{7}{11}$, $\frac{200}{14}$, $\frac{143}{3}$, $\frac{3}{22}$, &c., are improper fractions.

16. A Mixed Number is a number made up of a whole number and a fraction.

Thus, $16\frac{3}{4}$, $103\frac{1}{2}$, $14\frac{2}{3}$, $999\frac{1}{2}$, $6\frac{2}{3}$, $2\frac{1}{4}$, &c., are mixed numbers.

17. An Improper Fraction is always equal either to a whole number or to a mixed number. The following will exemplify an improper fraction, and its equivalent mixed number:



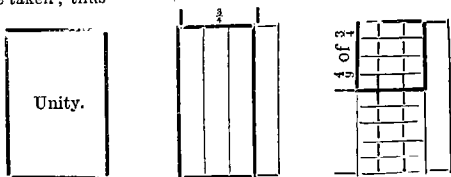
18. A Simple Fraction expresses one or more equal parts of unity.

Thus, $\frac{1}{2}$, $\frac{3}{8}$, $\frac{6}{6}$, $\frac{11}{15}$, $\frac{4}{5}$, $\frac{100}{3}$, &c., are simple fractions.

19. A Compound Fraction expresses one or more equal parts of a fraction; or, in other words, is a fraction of a fraction.

Thus, $\frac{2}{3}$ of $\frac{3}{4}$, $\frac{1}{2}$ of $\frac{7}{8}$ of $\frac{1}{10}$ of $\frac{9}{10}$ of $\frac{1}{2}$, &c., are compound fractions.

20. $\frac{4}{9}$ of $\frac{3}{4}$ means, not the four-ninths of unity, but the four-ninths of the three-fourths of unity;—that is, unity being divided into four parts, three of these are to be divided into nine parts and then four of these nine are to be taken; thus—

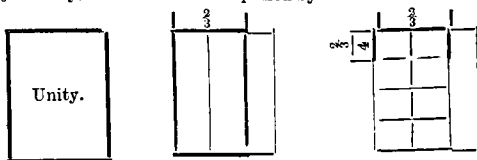


NOTE.—The word “of,” placed between the several parts of a compound fraction, is equal to and may be replaced by \times , the sign of multiplication.

21. A Complex Fraction is one having a fraction or a mixed number in its numerator or denominator, or in both.

Thus, $\frac{2}{\frac{3}{4}}$, $\frac{\frac{4}{7}}{7}$, $\frac{3}{\frac{7}{1}}$, $\frac{\frac{4}{5}}{\frac{1}{1}}$, $\frac{9\frac{1}{2}}{18\frac{2}{3}}$, $\frac{\frac{4}{7}}{21\frac{3}{5}}$, $\frac{6\frac{1}{2}}{5\frac{2}{3}}$, &c., are complex fractions.

NOTE. $\frac{3}{4}$ means, that we are to take the fourth part, not of unity, but of the $\frac{3}{4}$ of unity. This will be exemplified by—



22. Since fractions, like integers, are capable of being increased or diminished, they may be added, subtracted, &c.

23. Every integer may be considered as a fraction having unity for its denominator.

Thus, 13 may be written $1\frac{3}{1}$; $6\frac{4}{1}$; $29\frac{2}{1}$, &c.

REDUCTION OF FRACTIONS.

24. Since (Art. 8) multiplying both numerator and denominator by the same number does not alter the value of the fraction, we may reduce an integer to a fraction having any proposed denominator, by the following:—

RULE.

Write the integral number in the form of a fraction having 1 for its denominator. (Art. 23.)

And multiply both numerator and denominator of the resulting expression by the proposed denominator. (Art. 8.)

EXAMPLE 1.—Reduce 16 to a fraction having 11 for its denominator.

$$16 = 16 \times \frac{11}{11} = \frac{176}{11}.$$

EXAMPLE 2.—Reduce 173 to a fraction having 31 for its denominator.

$$173 = 173 \times \frac{31}{31} = \frac{5363}{31}.$$

EXERCISE.

- Reduce 29 to a fraction having 12 for its denominator.
Ans. $\frac{348}{12}$.
- Reduce 243 to a fraction having 3 for its denominator.
Ans. $\frac{729}{3}$.
- Reduce 7, 23, and 101 to fractions having 13 for denominator.
Ans. $\frac{91}{13}, \frac{299}{13}, \frac{1313}{13}$.
- Reduce 4, 37, 126, 73, and 1007 to fractions having 101 for denominator.
- Reduce 204, 7011, and 1999 to fractions having 207 for denominator.

25. Let it be required to reduce the mixed number $8\frac{7}{7}$ to an improper fraction.

$8\frac{7}{7}$ is equal to the whole number 8, and the fraction $\frac{7}{7}$, and by (Art. 24) $8 = \frac{56}{7}$, therefore $8\frac{7}{7} = \frac{56}{7} + \frac{7}{7} = \frac{63}{7}$.

Hence, to reduce a mixed number to an improper fraction, we deduce the following:—

RULE.

Multiply the whole number by the denominator of the fraction, to the product add the given numerator and place the sum over the given denominator.

EXAMPLE 8.—Reduce $73\frac{1}{9}$ to an improper fraction.

OPERATION. EXPLANATION.—We multiply the whole number, 73, by 9 and add in the numerator, 1. This gives us 658, which we write over the given denominator, 9, and the resulting fraction, $\frac{659}{9}$, is the improper fraction sought.

EXAMPLE 9.—Reduce $276\frac{1}{9}$ to an improper fraction

$$276\frac{1}{9} = \frac{276 \times 9 + 1}{9} = \frac{2485}{9} \text{ Ans.}$$

EXERCISE.

9. Reduce the mixed number, $73\frac{4}{19}$, $18\frac{4}{11}$, and $128\frac{6}{30}$ to improper fractions. *Ans.* $1391\frac{4}{19}$, $201\frac{4}{11}$, and $1478\frac{2}{5}$.

10. Reduce the mixed numbers $384\frac{5}{9}$, $673\frac{8}{13}$, $4792\frac{1}{26}$, and $568\frac{2}{26}$ to improper fractions.

Ans. $3451\frac{5}{9}$, $8757\frac{8}{13}$, $119801\frac{1}{26}$, and $14474\frac{2}{13}$.

26. Since every fraction indicates the division of the numerator by the denominator—to reduce an improper fraction to a mixed number, we have the following:—

RULE.

Divide the numerator by the denominator and the quotient will be the required mixed number.

EXAMPLE 11.—Reduce $204\frac{4}{7}$ to a mixed number.

$$204\frac{4}{7} = 204 \div 7 = 29\frac{1}{7} \text{ Ans.}$$

EXAMPLE 12.—Reduce $20047\frac{11}{11}$ to a mixed number.

$$20047 \div 11 = 1822\frac{5}{11} \text{ Ans.}$$

EXERCISE.

13. Reduce the improper fractions $4\frac{0}{13}$, $2\frac{0}{13}$, and $1\frac{0}{17}$ to mixed numbers. *Ans.* $31\frac{1}{13}$, $47\frac{8}{13}$, and $16\frac{1}{17}$

14. Reduce the improper fractions $2\frac{3}{12}$, $3\frac{0}{12}$, and $2\frac{0}{12}$ to mixed numbers. *Ans.* $88\frac{3}{12}$, $158\frac{4}{12}$, and 78 .

27. To reduce a fraction to its lowest terms:—

RULE.

Divide both terms by their greatest common measure.

This is simply dividing both terms by the same number—which does not affect the value of the fraction. (Art. 8.)

The greatest common measure may be found by (Art. 26, Sec. III.) or, very frequently, by inspection.

EXAMPLE 15.—Reduce $\frac{75}{9}$ to its lowest terms.

Greatest common measure = 25. Dividing both terms by 25; $\frac{75}{9} = \frac{3}{3}$ *Ans.*

EXAMPLE 16.—Reduce $\frac{126}{162}$ to its lowest terms.

Greatest common measure of 126 and 162 = 18.

Dividing both terms by 18 we get $\frac{126}{162} = \frac{7}{9}$ *Ans.*

EXERCISE.

17. Reduce $\frac{60}{250}$ to its lowest terms.

Ans. $\frac{1}{10}$.

18. Reduce $\frac{17378}{17378}$ to its lowest terms.

Ans. $\frac{1}{1}$.

19. Reduce $\frac{25850}{23712}$ and $\frac{570}{504}$ to their lowest terms. *Ans.* $\frac{1}{3}$ and $\frac{5}{8}$.

20. Reduce $\frac{2458}{4176}$, $\frac{510}{1728}$ and $\frac{3712}{68961}$ to their lowest terms.

Ans. $\frac{1}{24}$, $\frac{5}{27}$, and $\frac{500}{7323}$.

28. *Instead of dividing both terms by their greatest common measure we may divide both by any common measure. We thus reduce the fraction to lower terms, and, continuing the division as long as the terms have a common measure, we shall finally have reduced the fraction to its lowest terms.*

NOTE.—It is advisable to commit to memory the properties of numbers given in Art. 19, Sec. III, from XVIII to XXIV.

EXAMPLE 21.—Reduce $\frac{22480}{23416}$ to its lowest terms.

$$\begin{aligned} & \frac{22480}{23416} \text{ dividing by 10. (XX. of Art. 19, Sec. III.)} \\ &= \frac{2248}{2341.6} \text{ dividing by 8. (XXII. of Art. 19, Sec. III.)} \\ &= \frac{278.5}{292.7} \text{ dividing by 9. (XXIV. of Art. 19, Sec. III.)} \\ &= \frac{30.9}{60.4} \text{ dividing by 3. (XXIII. of Art. 19, Sec. III.)} \\ &= \frac{10.3}{20.1} \text{ Ans.} \end{aligned}$$

EXAMPLE 22.—Reduce $\frac{3605}{3617}$ to its lowest terms.

$$\begin{aligned} & \frac{3605}{3617} \text{ dividing by 5. (XIX. in Art. 19, Sec. III.)} \\ &= \frac{721}{723.4} \text{ dividing by 9. (XXIV. in Art. 19, Sec. III.)} \\ &= \frac{80.1}{80.4} \text{ dividing by 3. (XXIII. in Art. 19, Sec. III.)} \\ &= \frac{17}{29} \text{ Ans.} \end{aligned}$$

EXERCISE.

23. Reduce $\frac{720}{720}$ to its lowest terms. *Ans.* $\frac{1}{1}$.

24. Reduce $\frac{130550}{40000}$ to its lowest terms. *Ans.* $\frac{119}{40}$.

25. Reduce $\frac{330000}{330000}$ to its lowest terms. *Ans.* $\frac{1}{1}$.

26. Reduce $\frac{1380}{1380}$ to its lowest terms. *Ans.* $\frac{1}{1}$.

27. Reduce $\frac{28}{307}$, $\frac{540}{7143}$ and $\frac{17000}{17000}$ to their lowest terms. *Ans.* $\frac{1}{11}$, $\frac{183}{17}$ and $\frac{1}{1}$.

29. To reduce fractions of different denominators to equivalent fractions having the same denominator:—

RULE.

Multiply each numerator by all the denominators except its own, for a new numerator and all the denominators together for a new denominator.

This is merely multiplying both numerator and denominator of each fraction by the same quantity, viz: the product of all the other denominators, and consequently (Art. 8) it does not alter the value of the fraction.

EXAMPLE 28.—Reduce $\frac{3}{4}$, $\frac{7}{11}$, and $\frac{5}{9}$ to a common denominator.

$$\begin{aligned} 3 \times 11 \times 9 &= 297 = \text{1st numerator.} \\ 7 \times 4 \times 9 &= 252 = \text{2nd numerator.} \\ 5 \times 4 \times 11 &= 220 = \text{3rd numerator.} \\ 4 \times 11 \times 9 &= 396 = \text{common denominator.} \end{aligned}$$

Therefore the equivalent fractions are $\frac{297}{396}$, $\frac{252}{396}$, and $\frac{220}{396}$.

EXAMPLE 29.—Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{7}$, and $\frac{9}{11}$ to equivalent fractions having a common denominator.

$$\begin{aligned} 1 \times 5 \times 7 \times 11 &= 385 = \text{1st numerator.} \\ 3 \times 2 \times 7 \times 11 &= 462 = \text{2nd numerator.} \\ 4 \times 2 \times 5 \times 11 &= 440 = \text{3rd numerator.} \\ 9 \times 2 \times 5 \times 7 &= 630 = \text{4th numerator.} \\ 2 \times 5 \times 7 \times 11 &= 770 = \text{common denominator.} \end{aligned}$$

And the equivalent fractions are $\frac{385}{770}$, $\frac{462}{770}$, $\frac{440}{770}$ and $\frac{630}{770}$.

EXERCISE.

30. Reduce $\frac{2}{5}$, $\frac{5}{7}$, $\frac{8}{9}$, $\frac{3}{8}$, and $\frac{6}{11}$ to equivalent fractions having a common denominator.

$$\text{Ans. } \frac{11340}{28350}, \frac{20350}{28350}, \frac{25200}{28350}, \frac{17010}{28350}, \text{ and } \frac{7875}{28350}.$$

31. Reduce $\frac{2}{11}$, $\frac{1}{13}$, and $\frac{5}{17}$ to fractions having a common denominator.

$$\text{Ans. } \frac{1452}{2002}, \frac{154}{2002}, \frac{715}{2002}.$$

32. Reduce $\frac{9}{7}$, $\frac{1}{11}$, $\frac{6}{13}$, $\frac{4}{7}$, and $\frac{1}{2}$ to fractions having a common denominator.

$$\text{Ans. } \frac{12012}{14014}, \frac{5026}{14014}, \frac{5220}{14014}, \frac{8008}{14014}, \text{ and } \frac{7007}{14014}.$$

33. Reduce $\frac{6}{11}$, $\frac{4}{7}$, and $\frac{8}{13}$ to a common denominator.

$$\text{Ans. } \frac{316}{1001}, \frac{572}{1001}, \text{ and } \frac{616}{1001}.$$

34. Reduce $\frac{5}{6}$, $\frac{7}{7}$, $\frac{1}{8}$, and $\frac{2}{11}$ to a common denominator.

$$\text{Ans. } \frac{1925}{2310}, \frac{1320}{2310}, \frac{287}{2310}, \text{ and } \frac{420}{2310}.$$

35. Reduce $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, and $\frac{5}{7}$ to a common denominator.

$$\text{Ans. } \frac{105}{210}, \frac{140}{210}, \frac{126}{210}, \text{ and } \frac{150}{210}.$$

30. To reduce fractions to equivalent fractions having their *least* common denominator :—

RULE.

Find the least common multiple of all the denominators. (Art. 33, Sec. III.)

Multiply both terms of each fraction by the quotient obtained by dividing this least common multiple by the denominator of that fraction.

This is merely multiplying both terms by the same quantity, as in Art. 29.

EXAMPLE 36.—Reduce $\frac{1}{4}$, $\frac{7}{12}$, $\frac{5}{3}$, and $\frac{9}{16}$ to their least common denominator.

The least common multiple of 4, 12, 3, and 16, is 48.

Multiplying both terms of the 1st fraction by 12 (i.e. $\frac{48}{4}$) it becomes $\frac{12}{48}$.

“ “ “ 2nd “ by 4 (i.e. $\frac{48}{12}$) it becomes $\frac{28}{48}$.

“ “ “ 3rd “ by 16 (i.e. $\frac{48}{3}$) it becomes $\frac{32}{48}$.

“ “ “ 4th “ by 3 (i.e. $\frac{48}{16}$) it becomes $\frac{27}{48}$.

The equivalent fractions having their *least* common denominator, are therefore $\frac{12}{48}$, $\frac{28}{48}$, $\frac{32}{48}$, and $\frac{27}{48}$.

EXAMPLE 37.—Reduce $\frac{4}{5}$, $\frac{8}{11}$, $\frac{20}{20}$, $\frac{34}{44}$, $\frac{12}{27}$, and $\frac{1}{4}$ to their least common denominator.

The least common multiple of 5, 11, 20, 44, 55, and 4, is 220.

The multiplier for both terms of the first fraction is $\frac{220}{5} = 44$, for second, $\frac{220}{11} = 20$; for the third, $\frac{220}{20} = 11$; for the fourth, $\frac{220}{44} = 5$; for the fifth, $\frac{220}{55} = 4$; and for the sixth, $\frac{220}{4} = 55$.

Multiplying by these numbers, we obtain $\frac{176}{220}$, $\frac{160}{220}$, $\frac{319}{220}$, $\frac{155}{220}$, $\frac{76}{220}$, and $\frac{125}{220}$ for the required fractions.

EXERCISE.

38. Reduce $\frac{4}{5}$, $\frac{3}{8}$, $\frac{1}{3}$, and $\frac{1}{16}$ to their least common denominator,

Ans. $\frac{104}{120}$, $\frac{45}{120}$, $\frac{80}{120}$, $\frac{76}{120}$, and $\frac{76}{120}$.

39. Reduce $\frac{6}{11}$, $\frac{2}{3}$, $\frac{1}{7}$, $\frac{12}{77}$, and $\frac{19}{33}$ to their least common denominator.

Ans. $\frac{106}{231}$, $\frac{154}{231}$, $\frac{33}{231}$, $\frac{64}{231}$, and $\frac{133}{231}$.

40. Reduce $\frac{1}{2}$, $\frac{3}{3}$, $\frac{8}{6}$, $\frac{5}{3}$, $\frac{1}{10}$, $\frac{12}{16}$, $\frac{7}{10}$, and $\frac{3}{4}$ to their least common denominator.

Ans. $\frac{120}{240}$, $\frac{160}{240}$, $\frac{144}{240}$, $\frac{200}{240}$, $\frac{240}{240}$, $\frac{216}{240}$, $\frac{208}{240}$, $\frac{105}{240}$, and $\frac{144}{240}$.

41. Reduce $\frac{3}{8}$, $\frac{7}{10}$, $\frac{6}{25}$, $\frac{11}{30}$, $\frac{13}{45}$, and $\frac{23}{60}$ to their least common denominator.

Ans. $\frac{540}{900}$, $\frac{630}{900}$, $\frac{216}{900}$, $\frac{330}{900}$, $\frac{360}{900}$, and $\frac{345}{900}$.

42. Reduce $\frac{12}{20}$, $\frac{7}{30}$, $\frac{11}{40}$, and $\frac{1}{30}$ to their least common denominator.

Ans. $\frac{570}{600}$, $\frac{140}{600}$, $\frac{165}{600}$, and $\frac{135}{600}$.

43. Reduce $\frac{1}{2}$, $\frac{3}{3}$, $\frac{5}{6}$, $\frac{7}{8}$, $\frac{11}{12}$, $\frac{15}{16}$, and $\frac{23}{24}$ to their least common denominator.

Ans. $\frac{24}{24}$, $\frac{22}{24}$, $\frac{30}{24}$, $\frac{36}{24}$, $\frac{42}{24}$, $\frac{44}{24}$, and $\frac{46}{24}$.

44. Reduce $\frac{6}{7}$, $\frac{11}{12}$, $\frac{5}{15}$, $\frac{8}{27}$, $\frac{9}{35}$, and $\frac{1}{15}$ to their least common denominator.

Ans. $\frac{5400}{7560}$, $\frac{6930}{7560}$, $\frac{1008}{7560}$, $\frac{2240}{7560}$, $\frac{1344}{7560}$, and $\frac{3213}{7560}$.

45. Reduce $\frac{14}{15}$, $\frac{7}{5}$, $\frac{4}{3}$, $\frac{11}{12}$, $\frac{6}{17}$, $\frac{19}{20}$, $\frac{7}{9}$, and $\frac{8}{35}$ to their least common denominator.

Ans. $\frac{8928}{8928}$, $\frac{8028}{8928}$, $\frac{12320}{8928}$, $\frac{6170}{8928}$, $\frac{5040}{8928}$, $\frac{8778}{8928}$, $\frac{3230}{8928}$, and $\frac{7620}{8928}$.

31. Let it be required to reduce $\frac{12}{17}$ of $\frac{6}{11}$ to a simple fraction.

$\frac{12}{17}$ of $\frac{6}{11}$ means 12 times $\frac{1}{17}$ of $\frac{6}{11}$.

We get $\frac{1}{17}$ of $\frac{6}{11}$, i. e., divide $\frac{6}{11}$ by 17, when we multiply the denominator 11 by 17 (Art. 8). Therefore $\frac{1}{17}$ of $\frac{6}{11} = \frac{6}{11 \times 17}$, and to multiply this result by 12, we multiply the numerator, 6, by 12, (Art. 8.)

Therefore $\frac{12}{17}$ of $\frac{6}{11} = \frac{6 \times 12}{11 \times 17} = \frac{72}{187}$

Hence, to reduce a compound fraction to a simple one we deduce the following:—

RULE.

Multiply all the numerators together, for a new numerator, and all the denominators together for a new denominator.

EXAMPLE 46. Reduce $\frac{2}{3}$ of $\frac{4}{7}$ of $\frac{5}{9}$ to a simple fraction.

$$\frac{2}{3} \text{ of } \frac{4}{7} \text{ of } \frac{5}{9} = \frac{2 \times 4 \times 5}{3 \times 7 \times 9} = \frac{40}{189} \text{ Ans.}$$

NOTE.—In all cases the answer must be reduced to its lowest terms.

EXERCISE.

47. Reduce $\frac{4}{7}$ of $\frac{2}{3}$ of $\frac{5}{11}$ of $\frac{3}{2}$ to a simple fraction. *Ans.* $\frac{1}{11}$.
 48. Reduce $\frac{2}{3}$ of $\frac{4}{7}$ of $\frac{5}{11}$ of $\frac{3}{2}$ to a simple fraction. *Ans.* $\frac{3}{14}$.
 49. Reduce $\frac{2}{3}$ of $\frac{5}{11}$ of $\frac{3}{2}$ to a simple fraction. *Ans.* $\frac{5}{16}$.
 50. Reduce $\frac{2}{3}$ of $\frac{4}{7}$ of $\frac{5}{11}$ of $\frac{3}{2}$ to a simple fraction. *Ans.* $\frac{31}{64}$.

32. Since the several numerators of the compound fraction form the factors of the numerator of the simple fraction, and also the several denominators of the compound fraction, the factors of the denominator of the simple fraction, it follows (Art. 8) that,—

Before applying the rule in (Art. 31), we may cast out or cancel all the factors that are common to a numerator and a denominator of the compound fraction.

EXAMPLE 51.—Reduce $\frac{6}{11}$ of $\frac{4}{7}$ of $\frac{2}{5}$ of $\frac{22}{7}$ of $\frac{35}{16}$ to a simple fraction.

| STATEMENT. | CANCELLED. |
|---|--|
| $\frac{6}{11} \text{ of } \frac{4}{7} \text{ of } \frac{2}{5} \text{ of } \frac{22}{7} \text{ of } \frac{35}{16} = \frac{6 \times 4 \times 2 \times 22 \times 35}{11 \times 7 \times 5 \times 7 \times 16}$ | $= \frac{\overset{2}{\cancel{6}} \times \overset{2}{\cancel{4}} \times \overset{5}{\cancel{2}} \times \overset{7}{\cancel{22}} \times \overset{7}{\cancel{35}}}{\underset{3}{\cancel{11}} \times \underset{8}{\cancel{7}} \times \underset{8}{\cancel{5}} \times \underset{2}{\cancel{7}} \times \underset{2}{\cancel{16}}} = \frac{1}{3} \text{ Ans}$ |

Here 6 and 27 contain a common factor, 3, which is cast out, and these numbers thus reduced to 2 and 9. Next this 2 reduces 16 to 8, and the 9 is reduced to 3 by the third numerator, which is thus cancelled. Again, 11 cancels 11 (the first denominator) and reduces 22 to 2, and this 2 reduces the 8, before obtained from the 16, to 4. Next, this 4 is cancelled by the 4 in the numerator. Again, 7 cancels the 7 in the denominator and reduces the 35, in the numerator, to 5, and this 5 cancels the 5 in the denominator. All the numerators are now reduced to unity, as also all the denominators but the fourth, which is 3. The resulting fraction is therefore $\frac{1 \times 1 \times 1 \times 1}{1 \times 1 \times 1 \times 3} = \frac{1}{3}$ but this is simply $\frac{1}{3}$.

EXAMPLE 52.—Reduce $\frac{7}{11}$ of $\frac{4}{6}$ of $\frac{3}{5}$ of $\frac{55}{20}$ to a simple fraction.

| STATEMENT. | CANCELLED. |
|---|--|
| $\frac{7}{11} \text{ of } \frac{4}{6} \text{ of } \frac{3}{5} \text{ of } \frac{55}{20} = \frac{7 \times 4 \times 3 \times 55}{11 \times 6 \times 5 \times 20}$ | $= \frac{\overset{5}{\cancel{7}} \times \overset{5}{\cancel{4}} \times \overset{5}{\cancel{3}} \times \overset{11}{\cancel{55}}}{\underset{2}{\cancel{11}} \times \underset{5}{\cancel{6}} \times \underset{4}{\cancel{5}} \times \underset{4}{\cancel{20}}} = \frac{7}{2 \times 5} = \frac{7}{10} \text{ Ans.}$ |

NOTE.—If any of the terms of the compound fraction are whole or mixed numbers, they must be reduced to fractions (Arts. 23 and 25).

The process of cancelling exemplified above should always be adopted when possible.

EXERCISE.

53. Reduce $\frac{5}{9}$ of $\frac{6}{7}$ of $\frac{2}{3}$ of $\frac{3}{5}$ of $\frac{3}{10}$ to a simple fraction. *Ans.* $\frac{6}{84}$.

54. Reduce $\frac{2}{3}$ of $\frac{5}{9}$ of $\frac{1}{12}$ of $\frac{1}{11}$ of $\frac{1}{7}$ of $\frac{1}{3}$ to a simple fraction. *Ans.* $\frac{1}{882}$.

55. Reduce $\frac{2}{3}$ of $\frac{1}{11}$ of $5\frac{1}{2}$ to a simple fraction. *Ans.* $\frac{1}{11}$.

56. Reduce $\frac{1}{3}$ of $\frac{2}{3}$ of $\frac{1}{6}$ of $\frac{1}{6}$ of $\frac{1}{6}$ of $\frac{1}{6}$ of $2\frac{1}{5}$ to a simple fraction. *Ans.* $\frac{1}{180}$.

57. Reduce $\frac{3}{11}$ of $\frac{1}{7}$ of $\frac{2}{17}$ of $\frac{3}{4}$ of $\frac{3}{2}$ of $6\frac{1}{2}$ to a simple fraction. *Ans.* $\frac{3}{154}$.

58. Reduce $\frac{1}{4}$ of $\frac{3}{11}$ of 154 to a simple fraction. *Ans.* 24.

33. Let it be required to reduce the complex fraction $\frac{\frac{6}{7}}{\frac{3}{4}}$ to a simple fraction.

Since (Art. 8) we may multiply both numerator and denominator of a fraction by the same number, without altering its value—we may multiply both terms of the given fraction by $\frac{4}{3}$, i. e., by the denominator with its terms inverted, without altering its value.

$$\text{Therefore } \frac{\frac{6}{7}}{\frac{3}{4}} = \frac{\frac{6}{7} \times \frac{4}{3}}{\frac{3}{4} \times \frac{4}{3}} = \frac{\frac{6}{1} \times \frac{4}{3}}{1} = \frac{6}{1} \times \frac{4}{3} = \frac{6 \times 4}{7 \times 3}$$

Hence, to reduce a complex fraction to a simple one, we deduce the following:—

RULE.

Reduce the expression (Arts. 23 and 25) to the form of $\frac{\text{fraction}}{\text{fraction}}$;
i. e., reduce both numerator and denominator to simple fractions.

Then multiply the extremes or outside numbers together for a new numerator, and the means or intermediate numbers together for a new denominator.

EXAMPLE 59.—Reduce $\frac{4\frac{1}{2}}{7\frac{1}{11}}$ to a simple fraction.

$$\frac{4\frac{1}{2}}{7\frac{1}{11}} = \frac{\frac{9}{2}}{\frac{77}{11}} = \frac{9 \times 11}{2 \times 77} = \frac{99}{154} = 7\frac{1}{14} \text{ Ans.}$$

NOTE.—Factors that are common to one of the extremes and one of the means, are to be struck out or cancelled. (Art. 32.)

EXAMPLE 60.—Reduce $\frac{7\frac{1}{11}}{1\frac{1}{7}}$ to a simple fraction.

$$\frac{7\frac{1}{11}}{1\frac{1}{7}} = \frac{\frac{77}{11}}{\frac{8}{7}} = \frac{7 \times 7 \times 7}{8 \times 10} = \frac{63}{80} = 6\frac{3}{80} \text{ Ans.}$$

EXERCISE.

61. Reduce $\frac{\frac{1}{4}}{\frac{1}{2}\frac{7}{5}}$ to a simple fraction. *Ans.* $\frac{5}{7}$.
62. Reduce $\frac{\frac{1}{2}}{\frac{7}{1}\frac{7}{8}}$ to a simple fraction. *Ans.* $\frac{5}{25}$.
63. Reduce $\frac{15\frac{3}{5}}{7\frac{1}{5}}$ to a simple fraction. *Ans.* 2.
64. Reduce $\frac{11\frac{3}{8}}{12\frac{3}{8}}$, $\frac{3\frac{1}{2}}{9}$ and $\frac{7}{8}$ to simple fractions.
Ans. $\frac{1}{5}\frac{7}{8}$, $\frac{1}{3}\frac{3}{8}$, and $\frac{1}{4}\frac{8}{8}$.
65. Reduce $\frac{7\frac{7}{8}}{16\frac{1}{2}}$, $\frac{5\frac{7}{8}}{13\frac{5}{8}}$ and $\frac{2\frac{2}{3}}{3\frac{2}{3}}$ to simple fractions.
Ans. $\frac{1}{2}\frac{7}{8}$, $31\frac{1}{3}$, and $\frac{7}{10}$.
66. Reduce $\frac{16\frac{3}{4}}{11\frac{3}{4}}$, $\frac{6\frac{1}{2}}{13}$, $\frac{17}{18\frac{1}{2}}$, $\frac{21\frac{3}{4}}{10\frac{3}{4}}$, and $\frac{1}{4}\frac{3}{3}$ to simple fractions.
Ans. $1\frac{3}{4}$, $\frac{3}{8}\frac{1}{2}$, $\frac{5}{2}\frac{1}{2}$, $2\frac{1}{10}$, and $\frac{1}{4}\frac{3}{3}$.

34. A denominate fraction is a fraction of a denominate number.

Thus, $\frac{1}{2}$ of a lb., $\frac{7}{11}$ of a mile, $\frac{3}{8}$ of a day, &c., are denominate fractions.

35. Reduction of denominate fractions consists in changing them from one denomination to another without altering their values.

36. Let it be required to reduce $\frac{1}{4}$ of a pint to the fraction of a bushel.

Since 1 qt. = 2 pints, $\frac{1}{4}$ of a pint = $\frac{1}{2}$ of $\frac{1}{4}$ of a quart.

Also because 1 gal. = 4 qts. $\frac{1}{2}$ of a pint = $\frac{1}{4}$ of $\frac{1}{2}$ of $\frac{1}{4}$ of a gal.

Similarly $\frac{1}{4}$ of a pint = $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{4}$ of a bushel = $\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}$ bushel.

Hence to reduce a denominate fraction from a lower to a higher denomination, we deduce the following:—

RULE.

Take the number expressing how many of the given denomination are required to make one of the next higher; also the number expressing how many of this denomination are required to make one of the next higher again, and so on until the required denomination be reached.

Write the fractions formed by these numbers as denominators, with 1 as numerator and the given fraction in the form of a compound fraction, which reduce to a simple fraction. (Art. 32)

EXAMPLE 67.—Reduce $\frac{3}{11}$ of a minute to the fraction of a week.

Ans. $\frac{3}{11}$ of $\frac{1}{60}$ of $\frac{1}{24}$ of $\frac{1}{7}$ = $\frac{1}{36960}$ of a week.

EXAMPLE 68.—Reduce $\frac{6}{8}$ of a grain, troy, to the fraction of an ounce.

$\frac{6}{8}$ of $\frac{1}{24}$ of $\frac{1}{20}$ = $\frac{3}{5}$ of an oz. Troy.

EXERCISE.

69. Reduce $\frac{1}{2}$ of an oz. to the fraction of a pound, avoirdupois.

Ans. $\frac{1}{16}$ lb.

70. Reduce $\frac{1}{3}$ of $\frac{1}{3}$ of a penny to the fraction of a pound.

Ans. $\frac{1}{840}$.

71. Reduce $\frac{3}{4}$ of $8\frac{1}{2}$ days to the fraction of a week. *Ans.* $\frac{1}{8}$ wk.

72. Reduce $\frac{1}{7}$ of $16\frac{1}{2}$ nails to the fraction of an English ell.

Ans. $\frac{3}{280}$ E.e.

73. Reduce $\frac{1}{3}$ of $\frac{1}{11}$ of a yard to the fraction of a perch.

Ans. $\frac{2}{231}$ per.

74. Reduce $\frac{1}{3}$ of $\frac{1}{4}$ of $21\frac{1}{4}$ of a cord foot to the fraction of a cord.

Ans. $\frac{1}{28}$ cord.

75. Reduce $\frac{1}{9}$ of $\frac{1}{7}$ of $9\frac{1}{2}$ square perches to the fraction of an acre.

Ans. $\frac{1}{360}$ A.

37. Let it be required to reduce $\frac{1}{3}$ of a day to the fraction of a minute.

Since there are 24 hours in a day and 60 minutes in an hour;

$\frac{1}{3}$ of a day will be 24 times $\frac{1}{3}$ of an hour and 60 times 24 times $\frac{1}{3}$ of a minute; that is, $\frac{1}{3}$ of a day is equal to $\frac{1}{3} \times 24 \times 60$ of a minute

Therefore $\frac{1}{3}$ of a day = $\frac{1}{3}$ of 24^2 of 60 of a minute.

Hence, to reduce a denominate fraction from a higher to a lower denomination, we have the following:—

RULE.

Take the number expressing how many of the next lower denomination make one of the given denomination; also, the number, expressing how many of the next lower again make one of this denomination, and so on till the required denomination be reached.

Write the fractions formed by these numbers as numerators, with 1 as denominator, and the given fraction in the form of a compound fraction, which reduce to a simple fraction. (*Art.* 32)

EXAMPLE 76.—Reduce $\frac{1}{3}$ of a £ to the fraction of a penny.

$\frac{1}{3}$ of 20 of 12 = $\frac{400}{3}$ = 160 pence.

EXAMPLE 77.—Reduce $\frac{1}{3}$ of $\frac{1}{2}$ of $\frac{1}{11}$ of a furlong to the fraction of a foot.

$\frac{1}{3}$ of $\frac{1}{2}$ of $\frac{1}{11}$ of 40 of 100 of $\frac{1}{2}$ of 1 = 300 ft. *Ans.*

EXERCISE.

78. Reduce $\frac{1}{4}$ of a bushel to the fraction of a quart.
Ans. $\frac{4}{3}$ qt.
79. Reduce $\frac{2}{3}$ of a gal. to the fraction of $\frac{1}{2}$ of $\frac{2}{3}$ of a gill.
Ans. $\frac{16}{9}$.
80. Reduce $\frac{7}{9}$ of 2 pecks to the fraction of $\frac{1}{2}$ of $\frac{2}{3}$ of a pint.
Ans. $2\frac{2}{3}$.
81. Reduce $\frac{1}{2}$ of a lb. to the fraction of a scruple.
Ans. 24 scr.
82. Reduce $\frac{1}{8760}$ of $\frac{2}{3}$ of $\frac{1}{11}$ of $\frac{5}{7}$ of $\frac{2}{3}$ of a lb. avoirdupois to the fraction of a dram.
Ans. $\frac{1}{4378}$ dr.

38. To find the value of a denominate fraction in terms of a lower denomination:—

RULE.

Divide the numerator by the denominator according to the rule given in Art. 71, Sec. II.

This is only actually performing the work which the fraction indicates. (Art. 3.)

EXAMPLE 83.—What is the value of $\frac{1}{3}$ of a mile?

$$\begin{array}{r}
 11 \text{ miles} \div 3 \\
 \hline
 3 \overline{) 11} \text{ miles (6 fur. 30 per. } 4\frac{2}{3} \text{ yds. } \textit{Ans.} \\
 \underline{8} = \text{fur. in a mile.} \\
 \hline
 88 = \text{number of furlongs.} \\
 \underline{78} \\
 \hline
 10 \\
 \underline{9} = \text{perches in furlong.} \\
 \hline
 400 = \text{perches.} \\
 \underline{300} \\
 \hline
 10 \\
 \underline{5} = \text{yards in a perch.} \\
 \hline
 55 = \text{number of yards.} \\
 \underline{52} \\
 \hline
 3
 \end{array}$$

EXERCISE.

84. What is the value of $\frac{3}{11}$ of a bushel and also of $\frac{5}{7}$ of a lb. avoirdupois?
Ans. 1 pk. 0 gal. 0 qt. $1\frac{5}{11}$ pt. and 13 oz. $11\frac{2}{11}$ drams.
85. What is the value of $\frac{7}{13}$ of a yard of cloth?
Ans. 2 qrs. 0 na. $1\frac{5}{13}$ inches.
86. What is the value of $\frac{2}{3}$ of a lb., troy; and also of $\frac{1}{13}$ sq. mile?
Ans. 10 oz. 13 dwt. 8 grs.; and 62 acres, 1 rood, 8 sq. per. 4 sq. yds. 2 ft. $79\frac{1}{13}$ in.

87.—What is the value of $\frac{8}{9}$ of a furlong; and of $\frac{1}{4}$ of a £?

Ans. 35 rds. 3 yds. 0 ft. 2 in.; and 11s. 5 $\frac{1}{2}$ d.

39. Let it be required to reduce 2s. 7 $\frac{1}{2}$ d. to the fraction of £7 18s.

$\frac{2s. 7\frac{1}{2}d. = 127 \text{ farthings.}}{£7 18s = 7584 \text{ farthings.}}$ Therefore 2s. 7 $\frac{1}{2}$ d. = $\frac{127}{7584}$ of £7 18s.

Hence, to reduce one denominate number to the fraction of another, we deduce the following:—

RULE.

Reduce both quantities to the lowest denomination contained in either.

Then place that quantity which is to be the fraction of the other as numerator and the remaining quantity as denominator.

EXAMPLE 88.—Reduce 3 days 4 hours to the fraction of a week.

3 days 4 hours = 76 hours.

1 week = 168 hours.

And the required fraction is $\frac{76}{168} = \frac{19}{42}$ *Ans.*

EXAMPLE 89.—What fraction is 3 lb. 4 oz. 2 dr. 2 scr. 7 grs. of 63 lb. 4 oz. 7 dr. Apothecaries' weight?

3 lb. 4 oz. 2 dr. 2 scr. 7 grs. = 19367 grs.

63 lb. 4 oz. 7 dr. = 385220 grs.

And the fraction is $\frac{19367}{385220}$ *Ans.*

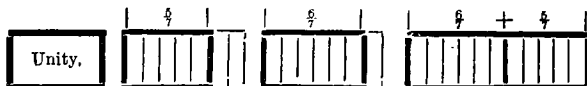
EXERCISE.

90. What fraction is 6 bush. 1 pk. 1 gal. 1 qt. of 50 bush.?
Ans. $\frac{411}{3200}$.
91. What fraction is 35 per. 9 ft. 2 in. of a furlong? *Ans.* $\frac{3}{8}$.
92. What fraction is 7 h. 12 m. of a day? *Ans.* $\frac{3}{10}$.
93. What fraction is 2 sq. yds. 2 ft. 120 in. of 3 sq. per. 13 $\frac{1}{4}$ yds. 1 ft. 72 in.? *Ans.* $\frac{1}{15}$.
94. What fraction is 7 oz. 7 dr. 2 scr. 14 grs. of 21 lbs. Apoth.?
Ans. $\frac{214}{2240}$.
95. Reduce 9 min. 48 sec. to the fraction of a day. *Ans.* $\frac{143}{14400}$.
96. Reduce 16 bush. 1 pk. 1 pt. to the fraction of 69 bush.
Ans. $\frac{347}{172}$.
97. Reduce 3 qrs. 3 $\frac{1}{2}$ na. to the fraction of an ell Eng. *Ans.* $\frac{31}{4}$.
98. What part of a lb. Troy is 13 dwt. 7 grs.? *Ans.* $\frac{319}{1760}$.
99. What part of 54 cords of wood is 4800 cubic feet? *Ans.* $\frac{3}{8}$.

ADDITION OF VULGAR FRACTIONS.

40. Addition of fractions is the process of finding a single fraction which shall express the value of all the fractions added.

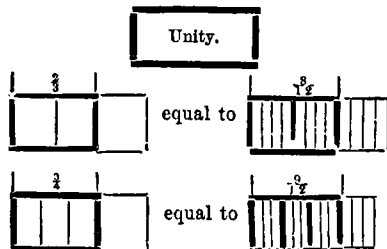
Addition may be illustrated as follows ;



41. In order that fractions may be added they must have a common denominator.

Thus $\frac{2}{3} + \frac{3}{4}$ make neither $\frac{5}{7}$ nor $\frac{5}{7}$; but if we reduce them to equivalent fractions having a common denominator, as $\frac{8}{12}$ and $\frac{9}{12}$, we are enabled to add them and thus obtain for their sum $\frac{17}{12}$.

These fractions, before and after they receive a common denominator, will be represented as follows :—



We have increased the number of the parts just as much as we have diminished their size.

42. For the addition of fractions we have therefore the following :—

RULE.

Reduce compound and complex fractions to simple ones, and all to a common denominator. (Arts. 29 and 30.)

Add all the numerators together, and beneath their sum place the common denominator.

Reduce the resulting fraction, when it is an improper fraction, to a mixed number. (Art. 26.)

NOTE.—If mixed numbers occur among the addends, the integral portions are to be added separately and their sum added to the sum of the fractions.

EXAMPLE 1.—Add together $\frac{4}{11}$, $\frac{3}{11}$, $\frac{2}{11}$, $\frac{7}{11}$ and $\frac{10}{11}$.

Here, since the fractions have already a common denominator, we have simply to add the numerators and place 11, the common denominator, beneath their sum.

$$\text{Thus } \frac{4}{11} + \frac{3}{11} + \frac{2}{11} + \frac{7}{11} + \frac{10}{11} = \frac{4+3+2+7+10}{11} = \frac{26}{11} = 2\frac{4}{11} \text{ Ans.}$$

EXAMPLE 2.—Add together $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$ and $\frac{1}{11}$.

These fractions reduced to their least common denominator by *Art. 30* become $\frac{28}{66}$, $\frac{48}{66}$, $\frac{52}{66}$, $\frac{55}{66}$, $\frac{6}{66}$.

$$\text{And } \frac{28}{66} + \frac{48}{66} + \frac{52}{66} + \frac{55}{66} + \frac{6}{66} = \frac{28+48+52+55+6}{66} = \frac{199}{66} = 3\frac{1}{66} \text{ Ans.}$$

EXAMPLE 3.—Add together $\frac{3}{4}$, $\frac{1}{2}$, $\frac{5}{8}$ and $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{5}{8}$ of $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{5}{8}$ of $\frac{1}{2}$.

$\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{5}{8}$ of $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{5}{8}$ is equal to $\frac{1}{16}$ (*Art. 31*).

The fractions to be added are therefore $\frac{3}{4} + \frac{1}{2} + \frac{5}{8} + \frac{1}{16}$.

These reduced to a common denominator (*Art. 20*), become

$$\frac{12}{16} + \frac{8}{16} + \frac{10}{16} + \frac{1}{16} = \frac{31}{16} \text{ Ans.}$$

EXAMPLE 4.—Add together $9\frac{1}{2}$, $11\frac{1}{3}$, $16\frac{2}{5}$, $43\frac{3}{7}$ and $\frac{4\frac{1}{2}}{7\frac{1}{2}}$.

Here the last fraction is a complex fraction and is equal to $\frac{9}{14}$.

$$9\frac{1}{2} + 11\frac{1}{3} + 16\frac{2}{5} + 43\frac{3}{7} + \frac{9}{14} = 9 + 11 + 16 + 43 + (\frac{1}{2} + \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{9}{14}).$$

$$\text{And } 9 + 11 + 16 + 43 = 79.$$

$$\text{Also } \frac{1}{2} + \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{9}{14} = \frac{105}{420} + \frac{140}{420} + \frac{168}{420} + \frac{180}{420} + \frac{270}{420} = \frac{863}{420} = 2\frac{2}{420}.$$

Therefore the sum of the given quantities is $79 + 2\frac{2}{420} = 82\frac{1}{210}$.

EXAMPLE 5.—Add together $\frac{8}{9}$, $\frac{3}{7}$ and $5\frac{3}{4}$.

Here adding the three fractions together we obtain $12\frac{43}{36}$ for their sum, to which we add the integral number 5 and thus obtain the entire sum $6\frac{3}{4}$.

EXERCISE.

6. Add together $\frac{1}{3}$, $\frac{1}{3}$ and $\frac{2}{3}$. *Ans.* $\frac{4}{3} = 1\frac{1}{3}$.

7. Add together $\frac{1}{12}$, $\frac{6}{12}$, $\frac{7}{12}$, $\frac{9}{12}$, $\frac{11}{12}$ and $\frac{5}{12}$. *Ans.* $\frac{39}{12} = 3\frac{3}{4}$.

8. Add together $4\frac{3}{7}$, $11\frac{4}{7}$, $16\frac{5}{7}$, $21\frac{6}{7}$ and $19\frac{7}{7}$. *Ans.* $71 + \frac{9}{7} = 73\frac{1}{7}$.

9. Add together $16\frac{2}{3}$, $11\frac{1}{3}$, $18\frac{4}{3}$, $17\frac{5}{3}$ and $11\frac{2}{3}$. *Ans.* $177\frac{1}{3}$.

10. Add together $4\frac{1}{4}$, $1\frac{1}{3}$ and $\frac{7}{12}$. *Ans.* $6\frac{2}{3}$.

11. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$ and $\frac{8}{9}$. *Ans.* $6\frac{4}{9}$.

12. Add together $\frac{2}{3}$, $\frac{5}{6}$ and $\frac{1}{2}$. *Ans.* $2\frac{1}{2}$.

13. Add together $\frac{4}{5}$, $\frac{5}{6}$, $\frac{6}{7}$, $\frac{7}{8}$ and $\frac{8}{9}$. *Ans.* $3\frac{1}{9}$.

14. Add together $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$ and $\frac{1}{7}$. *Ans.* $1\frac{1}{42}$.

15. Add together $16\frac{3}{11}$, $47\frac{2}{3}$, $21\frac{17}{33}$, $\frac{7}{18}$ and $19\frac{1}{2}$.
Ans. $104\frac{23}{66}$.
16. Add together $17\frac{1}{2}$, $43\frac{3}{7}$, $168\frac{4}{7}$, $207\frac{9}{21}$ and $506\frac{12}{28}$.
Ans. $943\frac{3}{7}$.
17. Add together $6\frac{3}{4}$, $11\frac{4}{7}$, $\frac{9}{36}$, $16\frac{7}{16}$, $\frac{1}{2}$, $\frac{5}{21}$ and $17\frac{11}{12}$.
Ans. $53\frac{123}{168}$.
18. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{7}{4}$ and $68\frac{1}{4}$.
Ans. $69\frac{11}{12}$.
19. Add together $173\frac{3}{12}$, $8\frac{5}{7}$ and $91\frac{11}{12}$.
Ans. $273\frac{23}{12}$.
20. Add together $11\frac{5}{6}$, $2\frac{2}{4}$, $3\frac{2}{3}$ and $4\frac{2}{3}$.
Ans. $13\frac{32}{6}$.
21. Add together $\frac{1}{8}$, $\frac{3}{12}$, $\frac{4}{8}$, $\frac{5}{24}$, $\frac{7}{6}$, $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{5}{8}$.
Ans. $3\frac{5}{8}$.
22. Add together $\frac{1}{7}$, $11\frac{1}{2}$, 18 , $26\frac{2}{7}$ and $79\frac{4}{7}$.
Ans. $142\frac{45}{7}$.
23. Add together $\frac{2}{3}$, $7\frac{2}{7}$ and $\frac{1}{5}$ of $\frac{3}{7}$ of $10\frac{1}{2}$.
Ans. $11\frac{74}{35}$.
24. Add together $\frac{41}{18}$, $\frac{1}{2}$ of $3\frac{3}{7}$ of $\frac{4}{5}$ of $2\frac{3}{4}$, and $20\frac{3}{7}$.
Ans. $15\frac{13}{18}$.
25. Add together $3\frac{5}{8}$, $11\frac{1}{6}$ and $14\frac{3}{8}$.
Ans. $29\frac{23}{24}$.
26. Add together $\frac{1}{2}$ of $\frac{3}{4}$, $\frac{2}{3}$ of $\frac{6}{7}$, $\frac{3}{5}$ of $\frac{7}{9}$, $\frac{2}{9}$ of $1\frac{7}{9}$ and $4\frac{1}{2}$ of $1\frac{1}{2}$ of $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{5}$.
Ans. $11\frac{28}{90}$.
27. Add together $41\frac{1}{2}$, $105\frac{2}{9}$, $300\frac{3}{4}$, $241\frac{1}{5}$ and $472\frac{1}{4}$.
Ans. $1161\frac{23}{36}$.
28. Add together $92\frac{5}{8}$, $37\frac{8}{9}$ and $7\frac{1}{6}$.
Ans. $137\frac{25}{72}$.
29. Add together $21\frac{1}{2}$, $35\frac{1}{6}$, $\frac{103}{24}$ and $\frac{2}{3}$ of $\frac{7}{8}$.
Ans. $61\frac{5}{8}$.
30. Add together $2\frac{3}{4}$ of $3\frac{2}{3}$, $1\frac{11}{16}$, $2\frac{1}{3}$ of $4\frac{1}{8}$ of $1\frac{3}{8}$, and $4\frac{3}{8}$ of $\frac{1}{5}$ of $2\frac{1}{8}$ of $1\frac{3}{7}$.
Ans. $34\frac{1133}{448}$.

43. In order to add denominate fractions they must not only have a common denominator, but they must be fractions of the same unit; i. e., must be of the same denomination.

Thus £3, 2s. and 1d. cannot be added together, as the result would be neither $\frac{2}{3}$ of a pound, $\frac{2}{3}$ of a shilling, nor $\frac{2}{3}$ of a penny.

But if we reduce them all to the fraction of a pound, or all to the fraction of a shilling, or all to the fraction of a penny, it is obvious that we may then add the resulting fractions, having first reduced them to a common denominator.

Hence, for the addition of denominate fractions, we have the following:—

RULE.

Reduce all the fractions to the same denomination (*Arts. 36 and 37*). Reduce the resulting fractions to a common denominator (*Arts. 29 and 30*). Add as in (*Art. 42*) and find the value of the resulting fraction (*Art. 38*).

EXAMPLE 31.—Add together $\frac{3}{4}$ of a day and $\frac{3}{4}$ of an hour.

$\frac{3}{4}$ of a day = $\frac{3}{4}$ of 24 = $18^h = 1^d \frac{6}{4}$ of an hour.

$$1^d \frac{6}{4} h. + \frac{3}{4} h. = \frac{1^d 1^h 2^m + 0^d 0^h 3^m}{2} = \frac{1^d 2^h 1^m}{2} = 5^d \frac{1}{2} h. = 5h. 45m. 42^s \frac{3}{4} \text{ sec.}$$

EXAMPLE 32.—Add together $\frac{7}{11}$ of a pound, $\frac{2}{3}$ of a shilling, and $\frac{3}{4}$ of a penny.

$\frac{7}{11}$ of a £ = $\frac{7}{11}$ of 20 of 12 = $16 \frac{8}{11}$ of a penny = $152 \frac{8}{11}$ pence.

$\frac{2}{3}$ of a shilling = $\frac{2}{3}$ of 12 = 8 of a penny = $4 \frac{2}{3}$ pence.

$$152 \frac{8}{11} + 4 \frac{2}{3} + \frac{3}{4} = 156 + \frac{280 + 308 + 165}{385} = 157 \frac{3}{385} \frac{8}{5} \text{ pence} = 13s. 1 \frac{3}{8} \frac{8}{5} d.$$

NOTE.—In place of proceeding as above, we may find the value of each fraction separately (Art. 38) and add the results.

EXAMPLE 33.—Add together $\frac{1}{4}$ of a bushel, $\frac{1}{2}$ of a peck, and $\frac{1}{11}$ of a gal.

$\frac{1}{4}$ of a bushel = 3 pks. 0 gal. 1 qt. $1 \frac{1}{6}$ pts.

$\frac{1}{2}$ of a peck = 1 gal. 3 qts.

$\frac{1}{11}$ of a gal. = 1 $\frac{5}{11}$ pts.

Sum = 1 bush. 0 pks. 0 gal. 1 qt. $0 \frac{2}{3} \frac{2}{5}$ pts. *Ans.*

EXERCISE.

34. What is the sum of $\frac{1}{11}$ lb. Apothecaries' weight, $\frac{3}{4}$ oz.

$\frac{1}{11}$ dr. and $\frac{6}{8}$ scr. *Ans.* 4 oz. 6 drs. 2 scrs. $18 \frac{1}{2} \frac{2}{3} \frac{2}{3}$ grs.

35. Add together $\frac{3}{5}$ yd. $\frac{1}{4}$ ell Eng. and $\frac{6}{7}$ qr.

Ans. 3 qrs. 3 na. $1 \frac{1}{4} \frac{3}{8}$ in.

36. Add together $\frac{1}{4}$ of a yard, $\frac{1}{4}$ of a foot, and $\frac{1}{4}$ of an in.

Ans. 7 inches.

37. What is the sum of $\frac{1}{11}$ of a mile, $\frac{1}{3}$ of a furlong, and

$\frac{2}{3}$ of a yard? *Ans.* 5 fur. 16 rds. 0 yds. 0 ft. $3 \frac{2}{3} \frac{2}{3}$ in.

38. What is the sum of $\frac{1}{4}$ wk. $\frac{1}{3}$ day $\frac{1}{5}$ h.?

Ans. 2 days 2 h. 12 m.

39. Add together $\frac{1}{7}$ £, $\frac{2}{3}$ s. and $\frac{1}{12}$ d. *Ans.* 3s. $1 \frac{2}{3} \frac{1}{4}$ d.

40. What is the sum of $\frac{5}{8}$ of 21s. $\frac{5}{8}$ of 5s. $\frac{5}{8}$ of £3 12s. 6d.

$\frac{1}{3}$ £, $\frac{7}{8}$ and $\frac{4}{5}$ d. ? *Ans.* £3 12s. $4 \frac{1}{3} \frac{2}{5}$ d.

SUBTRACTION OF VULGAR FRACTIONS.

44. Subtraction of vulgar fractions is the process of finding the difference between two fractions.

We have seen that before fractions can be added they must have a common denominator and that when denominate fractions are to be added they must be also of the same denomination, and this is manifestly the case also in the subtraction of fractions.

Hence, for the subtraction of fractions, we have the following:—

RULE.

Reduce compound and complex fractions to simple ones and all to the same denominator, if not already such.

Reduce both of the resulting fractions to a common denominator.

Subtract the numerator of the subtrahend from the numerator of the minuend, and beneath the difference write the common denominator.

NOTE.—In the case of mixed numbers it frequently happens that the fractional part of the subtrahend is greater than the fractional part of the minuend. When this occurs, instead of reducing both quantities to improper fractions and then applying the rule, it is much better to borrow *unity* from the integral part of the minuend and considering it as a fraction, having the common denominator, add it to the fractional part of the minuend. (See 3rd, 4th and 5th Examples below.)

EXAMPLE 1.—From $\frac{3}{7}$ take $\frac{2}{17}$.

$$\frac{3}{7} - \frac{2}{17} = \frac{51}{119} - \frac{14}{119} = \frac{37}{119} \text{ Ans.}$$

Here reducing $\frac{3}{7}$ and $\frac{2}{17}$ to a common denominator they become $\frac{51}{119}$ and $\frac{14}{119}$.

EXAMPLE 2.—From $\frac{2}{3}$ of $\frac{2}{7}$ of $\frac{1}{2}$ of 49 take $\frac{8\frac{3}{4}}{3\frac{1}{2}}$ of $\frac{1}{5}$ of $\frac{1}{3}$.

$$\text{Here } \frac{2}{3} \text{ of } \frac{2}{7} \text{ of } \frac{1}{2} \text{ of } 49 = \frac{2}{3}.$$

$$\text{And } \frac{8\frac{3}{4}}{3\frac{1}{2}} \text{ of } \frac{1}{5} \text{ of } \frac{1}{3} = \frac{1}{6}.$$

$$\text{And } \frac{2}{3} - \frac{1}{6} = \frac{4}{6} - \frac{1}{6} = \frac{3}{6} = \frac{1}{2} \text{ Ans.}$$

EXAMPLE 3.—From $192\frac{2}{11}$ take $16\frac{1}{6}$.

$\frac{2}{11}$ and $\frac{1}{6}$ reduced to a common denominator become $\frac{2}{33}$ and $\frac{11}{66}$.
 $192\frac{2}{11} - 16\frac{1}{6} = 192\frac{2}{33} - 16\frac{11}{66} = 191 + 1\frac{2}{33} - 16\frac{11}{66} = 191\frac{2}{33} - 16\frac{11}{66} = 175\frac{4}{66} \text{ Ans.}$

Here, since we cannot subtract $\frac{11}{66}$ from $\frac{2}{33}$, we have to borrow 1 from the integral part of the minuend, and considering it as $\frac{17}{33}$ add it to $\frac{2}{33}$. We thus reduce $192\frac{2}{33}$ to $191\frac{17}{33}$ and then make the subtraction.

EXAMPLE 4.—From $29\frac{2}{7}$ take $16\frac{1}{4}$.

$$29\frac{2}{7} - 16\frac{1}{4} = 29\frac{4}{14} - 16\frac{3.5}{14} = 28 + 1\frac{4}{14} - 16\frac{15}{14} = 28\frac{4}{14} - 16\frac{15}{14} = 12\frac{17}{14} \text{ Ans.}$$

EXAMPLE 5.—From $117\frac{3}{7}$ take $67\frac{1}{4}$.

$$117\frac{3}{7} - 67\frac{1}{4} = 117\frac{6}{14} - 67\frac{3.5}{14} = 116 + 1\frac{6}{14} - 67\frac{17.5}{14} = 116\frac{6}{14} - 67\frac{85}{14} = 49\frac{17}{14} \text{ Ans.}$$

EXAMPLE 6.—What is the difference between $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{4}{5}$ of 2 $\frac{1}{2}$ days and $\frac{2}{3}$ of $\frac{1}{4}$ of $\frac{5}{6}$ of 5 $\frac{1}{2}$ hours?

$\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{4}{5}$ of 2 $\frac{1}{2}$ days = $\frac{4}{5}$ of a day = $\frac{4}{5}$ of 24 of an hour = $17\frac{2}{5}$ hours =

$17\frac{4}{5}$ hours; and $\frac{2}{3}$ of $\frac{1}{4}$ of $\frac{5}{6}$ of 5 $\frac{1}{2}$ hours = $\frac{5}{6}$ hours = $1\frac{1}{2}$ hour,

$$\text{And } 17\frac{4}{5} \text{ h.} - 1\frac{1}{2} \text{ h.} = 17\frac{4}{5} - 1\frac{3}{6} = 16\frac{4}{6} \text{ hours. Ans.}$$

EXERCISE.

7. From $\frac{3}{4}$ take $\frac{7}{10}$. *Ans.* $\frac{2}{5}$.
8. From $\frac{906}{1496} + \frac{1}{17}$ of $\frac{3}{4}$ of $\frac{1}{11}$ take $\frac{8\frac{3}{4}}{6\frac{1}{11}}$. *Ans.* 0.
9. From $982\frac{1}{7}$ take $29\frac{1}{8}$. *Ans.* $952\frac{1}{14} + \frac{7}{8}$.
10. What is the difference between $69\frac{1}{11}$ and $18\frac{1}{10}$? *Ans.* $50\frac{1}{11} + \frac{1}{10}$.
11. What is the difference between $100\frac{1}{2}$ and $9\frac{5}{8}$? *Ans.* $90\frac{1}{8}$.
12. What is the difference between $6\frac{1}{2}$ and $\frac{1}{2}$ of $9\frac{1}{2}$? *Ans.* $1\frac{1}{2}$.
13. From $611\frac{1}{11}$ take $610\frac{1}{11}$. *Ans.* $\frac{87\frac{1}{11}}{100}$.
14. From $\frac{2}{3}$ of 2 take $\frac{2}{3}$ of $\frac{1}{2} + \frac{1}{4}$. *Ans.* $\frac{2}{3}$.
15. From $\frac{2}{3}$ of a lb. avoirdupois take $\frac{2}{3}$ of a dram. *Ans.* 10 oz. $9\frac{1}{2}$ drs.
16. What is the difference between $24\frac{1}{4}$ and $21\frac{1}{11}$? *Ans.* $2\frac{1}{44}$.
17. What is the difference between $\frac{2}{3}$ of a mile, and $\frac{1}{11}$ of a furlong? *Ans.* 1 fur. 5 rd. 3 yds. 1 ft. 10. in.
18. Find the value of $\frac{2}{3}$ of $1\frac{3}{4} - \frac{1}{8}$ of $28\frac{1}{2}$. *Ans.* $5\frac{1}{2}$.
19. Find the value of $12\frac{3}{4} + \frac{1}{2}$ of $\frac{2}{3}$ of $8\frac{1}{2}$ of $\frac{10\frac{3}{8}}{6\frac{2}{3}} - \frac{17\frac{1}{11}}{13\frac{1}{4}}$. *Ans.* $2\frac{2}{3}$.
20. Find the value of $3\frac{1}{2} + 8\frac{1}{2} - 3\frac{3}{10} - 2\frac{5}{8} + 5\frac{1}{2} + 6\frac{1}{2} - 16\frac{1}{2}$. *Ans.* $\frac{1}{10}$.
21. From $\frac{1}{11}$ of an acre take $\frac{1}{8}$ of a perch. *Ans.* 1 rood 17 p. 22 yds. 2 ft. 108 in.
22. From $16\frac{1}{2}$ take $9\frac{1}{2}$, and from $169\frac{1}{11}$ take $83\frac{1}{11}$. *Ans.* $6\frac{5}{11}$ and $85\frac{6}{11}$.

MULTIPLICATION OF VULGAR FRACTIONS.

45. Let it be required to multiply $\frac{3}{11}$ by $\frac{7}{8}$.

Here we are required to multiply $\frac{3}{11}$ by $\frac{7}{8}$, that is by $\frac{1}{8}$ of 7.

Now if we multiply $\frac{3}{11}$ by 7 we shall have multiplied by a quantity 8 times too great, and the product will be 8 times too great.

If, therefore, we multiply $\frac{3}{11}$ by 7 we shall have to divide the result by 8 in order to get the product of $\frac{3}{11} \times \frac{7}{8}$.

But (Art. 8) we multiply $\frac{3}{11}$ by 7, when we multiply the numerator by 7 and we divide the result by 8 when we multiply the denominator by 8.

Therefore, $\frac{3}{11} \times \frac{7}{8} = \frac{3 \times 7}{11 \times 8}$, that is to multiply fractions together, we multiply the numerators together for a new numerator, and the denominators together for a new denominator.

Hence, for the multiplication of Vulgar Fractions we deduce the following:—

RULE.

Reduce compound and complex fractions to simple ones (Arts. 32 and 33) and whole and mixed numbers to improper fractions (Arts. 23 and 25.)

Cancel any factors that are common to a numerator and a denominator of the resulting fractions (Art. 32).

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Multiply all the reduced numerators together for a new numerator, and all the reduced denominators together for a new denominator.

Reduce the result, if necessary, to a mixed number.

EXAMPLE 1.—Multiply $\frac{2}{3}$ by $\frac{1}{7}$.

$$\frac{2}{3} \times \frac{1}{7} = \frac{2 \times 1}{3 \times 7} = \frac{2}{21} \text{ Ans.}$$

Here we cancel the first denominator and reduce the second numerator to 3.

EXAMPLE 2.—Multiply together $\frac{7}{11}$, $\frac{4}{5}$, $3\frac{1}{2}$ and $\frac{5}{8}$.

STATEMENT.

CANCELLED.

$$\frac{7}{11} \times \frac{4}{5} \times \frac{3}{2} \times \frac{5}{8} = \frac{7}{11} \times \frac{4}{5} \times \frac{7}{2} \times \frac{5}{8} = \frac{1}{1} \text{ Ans.}$$

EXAMPLE 3.—Multiply together $\frac{4}{5}$, $\frac{3}{11}$, $6\frac{2}{7}$, $9\frac{3}{8}$, $2\frac{1}{2}$ and 63.

STATEMENT.

$$\frac{4}{5} \times \frac{3}{11} \times \frac{4}{7} \times \frac{9}{8} \times \frac{5}{2} \times 63$$

CANCELLED.

$$\frac{4}{5} \times \frac{3}{11} \times \frac{4}{7} \times \frac{48}{8} \times \frac{5}{2} \times \frac{63}{1} = \frac{2 \times 3 \times 4 \times 48}{1} = 1152 \text{ Ans.}$$

EXAMPLE 4.—Multiply together $\frac{1}{179}$, $18\frac{7}{11}$, $9\frac{3}{8}$, $\frac{1}{2}$ of $\frac{3}{4}$ of 7, and $\frac{2}{3}$ of $\frac{1}{4}$ of 25.

STATEMENT.

$$\frac{1}{179} \times \frac{205}{11} \times \frac{48}{8} \times \frac{21}{8} \times \frac{165}{4}$$

CANCELLED.

$$\frac{1}{179} \times \frac{205}{11} \times \frac{48}{8} \times \frac{21}{8} \times \frac{165}{4} = \frac{205 \times 3 \times 3 \times 3}{179} = \frac{5535}{179} = 30\frac{165}{179} \text{ Ans.}$$

EXAMPLE 5.—Multiply together $\frac{7}{9}$, $3\frac{4}{11}$, $4\frac{1}{2}$, $\frac{3}{5}$, $6\frac{1}{7}$ and $5\frac{3}{8}$.

STATEMENT.

$$\frac{7}{9} \times \frac{47}{11} \times \frac{9}{2} \times \frac{3}{5} \times \frac{43}{7} \times \frac{77}{8}$$

CANCELLED.

$$\frac{7}{9} \times \frac{247}{81} \times \frac{9}{2} \times \frac{3}{5} \times \frac{43}{7} \times \frac{77}{15} = \frac{247 \times 43 \times 77}{81 \times 5 \times 15} = \frac{817817}{6075} = 134\frac{3347}{6075}$$

EXERCISE.

6. What is the product of $\frac{1}{7} \times \frac{5}{6}$. *Ans.* $\frac{5}{42}$.
 7. What is the product of $\frac{5}{6} \times \frac{1}{2}$? *Ans.* $\frac{5}{12}$.
 8. What is the product of $\frac{1}{5} \times \frac{2}{3}$? *Ans.* $\frac{2}{15}$.
 9. Multiply together $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{1}{5}$. *Ans.* $\frac{1}{15}$.
 10. Multiply together 14, $15\frac{1}{6}$ and $3\frac{5}{6}$. *Ans.* 749 $\frac{1}{2}$.
 11. Multiply together $\frac{2}{3}$, $8\frac{1}{2}$, $\frac{1}{7}$ and $\frac{1}{12}$. *Ans.* $5\frac{1}{42}$.
 12. Required the product of $\frac{1}{6}$, $\frac{6}{10}$, $\frac{2}{7}$, $1\frac{1}{3}$ and $\frac{5}{8}$. *Ans.* $\frac{1}{112}$.
 13. Required the product of $\frac{2}{3}$, $\frac{1}{4}$, $\frac{5}{6}$, 21, $\frac{3}{4}$ and 5. *Ans.* 13 $\frac{1}{2}$.
 14. Required the product of $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and 209. *Ans.* $9\frac{3}{8}$.
 15. Find the value of $6\frac{1}{2} \times 11\frac{3}{4} \times 10\frac{1}{2} \times \frac{1}{5} \times \frac{7}{6}$ of $\frac{1}{90}$. *Ans.* $\frac{1}{11}$.
 16. Find the value of $\frac{1}{2}$ of $\frac{1}{11}$ of $\frac{1}{10}$ of $77 \times \frac{2}{3}$ of $\frac{3}{5}$ of $91 \times 6\frac{2}{3}$. *Ans.* 1127 $\frac{1}{2}$.

17. Multiply together $\frac{1}{8}$, $\frac{2}{9}$, $7\frac{1}{2}$, $4\frac{3}{8}$, $\frac{2}{7}$, and $1\frac{1}{2}$. *Ans.* $\frac{1}{107}$.
 18. Multiply $\frac{1}{4}$ of 8 by $\frac{2}{7}$ of 19. *Ans.* 10 $\frac{2}{7}$.
 19. Multiply $\frac{1}{10}$ of 7 by $\frac{1}{11}$ of $87\frac{3}{4}$. *Ans.* 403 $\frac{1}{4}$.
 20. Find the value of $6\frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4}$. *Ans.* $2\frac{1}{24}$.
 21. Find the value of $3\frac{1}{2} \times 4\frac{1}{2} \times 15$. *Ans.* 268 $\frac{1}{2}$.
 22. Multiply $\frac{1}{4}$ of $8\frac{1}{2}$ of $\frac{1}{10}$ of $9\frac{1}{2}$ by $8\frac{1}{2} \times \frac{1}{4}$ of $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{1}{4}$ of $1\frac{1}{2}$ of $1\frac{1}{2}$. *Ans.* $4729\frac{1}{2}$.

23. Find the value of $\frac{27}{37\frac{1}{2}} \times \frac{87\frac{2}{3}}{98\frac{1}{8}} \times \frac{7}{24} \times \frac{81\frac{6}{7}}{128}$ *Ans.* $\frac{1}{55}$.

24. Multiply $8\frac{1}{7}$ by $\frac{1}{2}$ of $\frac{2}{3}$ of $1\frac{1}{2}$. *Ans.* $5\frac{1}{4}$.

25. Find the value of $\frac{75\frac{3}{4}}{6\frac{1}{11}} \times \frac{3}{11}$ of $8\frac{1}{2} \times \frac{1}{16}$ of 28 $\frac{7}{15} \times \frac{1}{9} \times 14\frac{1}{2} \times \frac{1}{121}$

$\times \frac{4}{5\frac{1}{2}} \times \frac{5}{9}$. *Ans.* $17\frac{1}{2}$.

46. To multiply an integral denominate number by a fraction, we have the following:—

RULE.

Multiply the denominate number by the numerator of the fraction and divide the result by the denominator.

NOTE.—This is merely considering the denominate number as a fraction having 1 for its denominator (Art. 23), and applying the preceding rule.

EXAMPLE 26.—How much is $\frac{1}{3}$ of \$129.63?

$$\frac{1}{3} \text{ of } \$129.63 = \frac{\$129.63 \times 1}{3} = \frac{\$518.52}{9} = \$57.61\frac{1}{3}. \text{ Ans.}$$

EXAMPLE 27.—How much is $\frac{1}{7}$ of $\frac{1}{2}$ of 10 lb. 6 oz. 4 dr. Avoir.?

$$\frac{1}{7} \text{ of } \frac{1}{2} \text{ of } 10 \text{ lb. } 6 \text{ oz. } 4 \text{ dr.} = \frac{7 \text{ of } 10 \text{ lb. } 6 \text{ oz. } 4 \text{ dr.} \times 7}{28} = 3 \text{ lbs. } 4 \text{ oz. } 14\frac{1}{7} \text{ drams. Ans.}$$

EXERCISE.

28. How much is $1\frac{7}{10}$ of 4 days 5 h. ? *Ans.* 5 days 38 m. 20 sec.
 29. How much is $\frac{1}{3}$ of £29 ? *Ans.* £8 19s. 6 $\frac{2}{3}$ d.
 30. How much is $\frac{1}{5}$ of 186 acres 3 roods ? *Ans.* 145 acres 1 rood.
 31. How much is $\frac{1}{4}$ of $\frac{2}{3}$ of $\frac{1}{10}$ of $23\frac{1}{2}$ times 24 h. 30 m. ?
Ans. 1 hour 38 min.
 32. How much is $\frac{2}{3}$ of $\frac{1}{2}$ of $\frac{1}{6}$ of $\frac{1}{3}$ of 33 bush. 2 pk. 1 gal. ?
Ans. 2 bush. 2 pk. 0 gal. 3 qt. $1\frac{1}{2}$ pt.

47. From the principles already established, it is evident that—

1st. When the multiplier is less than unity, the product is less than the multiplicand.

2nd. To multiply a fraction by a whole number, we may either multiply the numerator of the fraction or divide the denominator by that number. (Art. 8.)

3rd. To multiply a whole number by any fraction having *unity* for its numerator, we simply divide the whole number by the denominator.

Thus, to multiply by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{7}$, $\frac{1}{11}$, &c., we divide by 2, 3, 4, 7, 11, &c.

4th. When multiplying by a mixed number of which the fractional part has *unity* for its numerator, it is better to multiply by the integral part of the multiplier first and then by the fractional part, afterwards adding the two partial products together.

DIVISION OF VULGAR FRACTIONS.

48. Let it be required to divide $\frac{3}{7}$ by $\frac{5}{11}$.

Here we are required to divide $\frac{3}{7}$ by $\frac{5}{11}$ that is, by $\frac{11}{5}$ of 5.

Now if we divide $\frac{3}{7}$ by 5, we use a divisor 11 times too great, and the quotient is 11 times less than the required quotient.

Therefore, to obtain the correct quotient of $\frac{3}{7} \div \frac{5}{11}$, after dividing $\frac{3}{7}$ by 5, we shall have to multiply the result by 11.

But (Art. 8) we divide the fraction $\frac{3}{7}$ by 5 when we multiply the denominator 7 by 5, and we multiply the result by 11 when we multiply the numerator 3 by 11.

Therefore $\frac{3}{7} \div \frac{5}{11} = \frac{3 \times 11}{7 \times 5} = 3 \times \frac{11}{5}$ —dividend \times divisor with its terms inverted.

Hence for the division of fractions we have the following:—

RULE.

Reduce compound and complex fractions to simple ones; whole and mixed numbers to improper fractions.

Invert the terms of the divisor and proceed as in multiplication.

In addition to the foregoing analysis, the following may be given as a proof of the truth of this rule.

$\frac{3}{7} \div \frac{5}{11} = \frac{3}{11}$ because the dividend of any question in division may be made the numerator and the divisor the denominator of a fraction.

Now since we may multiply both terms of the fraction $\frac{3}{11}$ by any number we may multiply them by $\frac{1}{5}$, i. e., the denominator with its terms inverted.

Therefore $\frac{3}{11} = \frac{3 \times \frac{1}{5}}{\frac{11}{5} \times \frac{1}{5}} = \frac{3 \times \frac{1}{5}}{1}$ (because $\frac{11}{5} \times \frac{1}{5} = 1$) $= \frac{3}{5} \times \frac{1}{5}$; whence the truth of the rule.

EXAMPLE 1.—Divide $\frac{3}{11}$ by $\frac{5}{11}$.

$$\frac{3}{11} \div \frac{5}{11} = \frac{3}{5} \times \frac{11}{11} = \frac{3}{5} \text{ Ans.}$$

EXAMPLE 2.—Divide $\frac{1}{2}$ of $\frac{7}{11}$ by $\frac{1}{11}$ of $8\frac{1}{2}$.

$$\frac{1}{2} \text{ of } \frac{7}{11} \div \frac{1}{11} \text{ of } 8\frac{1}{2} = \frac{7}{22} \div \frac{1}{22} = \frac{7}{22} \times \frac{22}{1} = \frac{7}{1} = 7 \text{ Ans.}$$

EXAMPLE 3.—Divide $8\frac{1}{2}$ by $3\frac{1}{11}$.

$$8\frac{1}{2} \div 3\frac{1}{11} = \frac{18}{2} \div \frac{34}{11} = \frac{18}{2} \times \frac{11}{34} = \frac{9}{1} \times \frac{11}{17} = \frac{99}{17} = 5\frac{14}{17} \text{ Ans.}$$

EXAMPLE 4.—Divide $\frac{3}{11}$ of $\frac{4}{11}$ of $\frac{8\frac{1}{2}}{11} \times 3\frac{1}{2}$ by $\frac{1}{11}$ of $\frac{9\frac{3}{4}}{8\frac{1}{2}} \times 4\frac{1}{2}$.

STATEMENT.

TERMS OF DIVISOR INVERTED.

$$\frac{3}{11} \times \frac{4}{11} \times \frac{32\frac{1}{2}}{11} \times 2\frac{1}{2} \div \frac{1}{11} \times \frac{36\frac{3}{4}}{8\frac{1}{2}} \times \frac{9\frac{3}{4}}{8\frac{1}{2}} = \frac{3}{11} \times \frac{4}{11} \times \frac{32\frac{1}{2}}{11} \times \frac{2\frac{1}{2}}{1} \times \frac{11}{36\frac{3}{4}} \times \frac{11}{9\frac{3}{4}} \times \frac{11}{8\frac{1}{2}} \times \frac{8\frac{1}{2}}{9\frac{3}{4}} \times \frac{8\frac{1}{2}}{4\frac{1}{2}}$$

CANCELLED.

$$= \frac{3}{11} \times \frac{4}{11} \times \frac{32\frac{1}{2}}{11} \times \frac{2\frac{1}{2}}{1} \times \frac{11}{36\frac{3}{4}} \times \frac{11}{9\frac{3}{4}} \times \frac{11}{8\frac{1}{2}} \times \frac{8\frac{1}{2}}{9\frac{3}{4}} \times \frac{8\frac{1}{2}}{4\frac{1}{2}} = \frac{35}{2 \times 3} = \frac{35}{6} = 5\frac{5}{6} \text{ Ans.}$$

EXERCISE.

5. Divide $\frac{1}{2}$ of $\frac{3}{4}$ by $\frac{1}{2}$ of $8\frac{1}{2}$. Ans. $1\frac{3}{7}$.
6. Divide $\frac{1}{2}$ by $1\frac{1}{2}$ and divide the result by $\frac{1}{11}$. Ans. $\frac{2}{3}$.
7. Divide $82\frac{1}{11}$ by $26\frac{1}{11}$. Ans. $3\frac{20\frac{6}{11}}{11}$.
8. Divide $2\frac{1}{2}$ by $\frac{1}{2} + \frac{1}{3}$. Ans. $1\frac{3}{4}$.
9. Divide $1\frac{1}{2}$ by $\frac{1}{4}$ of $2\frac{1}{2}$ of 16 of $8\frac{1}{2}$ of $\frac{1}{10}$. Ans. $2\frac{1}{10}$.
10. Divide $2\frac{1}{2}$ by $(\frac{1}{3} \div \frac{3}{12} \text{ of } 9)$. Ans. $7\frac{1}{10}$.
11. Divide $48\frac{1}{2}$ by $\frac{3}{4} + \frac{1}{2}$ of 6 . Ans. $19\frac{3}{4}$.
12. Divide $6\frac{1}{2}$ by $\frac{3}{4}$ of $\frac{1}{10} + \frac{1}{11}$. Ans. $6\frac{1}{11}$.

13. Divide $4\frac{1}{2}$ of $3\frac{1}{2}$ by $2\frac{1}{2}$ of $6\frac{1}{2}$. *Ans.* $1\frac{1}{5}$.
14. Divide $\frac{7\frac{1}{2}}{11\frac{3}{4}}$ by $\frac{3}{4}$. *Ans.* $6\frac{1}{11}\frac{5}{8}$.
15. Divide $\frac{5}{9}$ of $7\frac{3}{7}$ by $\frac{1}{7}$ of $17\frac{3}{4}$. *Ans.* $3\frac{1}{4}\frac{5}{8}$.
16. Divide $1\frac{1}{2}$ of $1\frac{1}{3}$ of $\frac{1}{2}$ of $1\frac{1}{2}$ by $\frac{2}{5}$ of $\frac{3}{5}$ of $\frac{1}{2}$ of 5. *Ans.* $3\frac{1}{2}\frac{1}{4}$.
17. Divide $\frac{1\frac{1}{2}}{4\frac{1}{2}}$ by $\frac{2\frac{1}{2}}{2\frac{1}{4}}$. *Ans.* $\frac{1}{4}$.
18. Divide $\frac{3}{5}$ by $\frac{4\frac{1}{2}}{17\frac{1}{4}}$. *Ans.* $\frac{1}{4}$.
19. Divide $14\frac{1}{2}$ of $\frac{1}{3}$ by $\frac{2}{3}$ of $8\frac{1}{3}$ of $\frac{6\frac{1}{2}}{19\frac{3}{4}}$. *Ans.* $1\frac{2}{3}\frac{63}{89}$.
20. Divide $15\frac{1}{2}$ of $\frac{7}{5}$ of $\frac{7}{3}$ of $\frac{7}{3}$ by $\frac{4\frac{5}{6}}{7}$ of $\frac{3}{4}$ of $\frac{7}{3}$ of $\frac{2\frac{1}{2}}{4}$. *Ans.* $28\frac{7}{2}\frac{3}{5}$.

49. To divide an integral denominate number by a fraction :—

RULE.

Multiply it by the denominator and divide the result by the numerator of the fraction.

NOTE.—This is, in effect, merely considering the denominate number as a fraction having 1 for its denominator (Art. 23) and applying the foregoing rule.

EXAMPLE 21.—Divide 6 days 17 hours 11 minutes by $\frac{5}{7}$.
 $6 \text{ days } 17 \text{ h. } 11 \text{ m.} \div \frac{5}{7} = 6 \text{ days } 17 \text{ h. } 11 \text{ m.} \times \frac{11}{5} = 6 \text{ days } 17 \text{ h. } 11 \text{ m.} \times \frac{11}{5}$
Ans. 14 days 19h. 36m. 12 sec.

EXERCISE.

22. Divide £8 14s. 6½d. by $\frac{1\frac{3}{4}}{1\frac{3}{4}}$. *Ans.* £8 8s. 5½d.
23. Divide 1m. 5 fur. 91 yds. 2 feet by $2\frac{1}{3}$ of $1\frac{2}{7}$.
Ans. 2 fur. 124 yds. 2 ft.
24. Divide 3 acres, 3 roods and 3 perches by $\frac{3}{4}$.
Ans. 6 acres 1 rood 5 per.
25. Divide £7 16s. 2d. by $\frac{4}{5}$. *Ans.* £17 11s. 4½d.

50. To reduce a fraction having a complex fraction in its numerator or denominator or both to a simple fraction we have simply to apply as often as necessary the rule given in Art. 33.

NOTE.—Particular attention must be paid to the relative length and heaviness of the separating lines as they determine the various numerators and denominators.

EXAMPLE 26.—Simplify

$$\frac{3\frac{1}{2}}{\frac{1}{5}} \div \frac{7\frac{1}{2}}{3\frac{1}{4}} \div \frac{5}{9}$$

OPERATION.

$$\frac{3\frac{1}{2}}{\frac{1}{5}} \left. \vphantom{\frac{3\frac{1}{2}}{\frac{1}{5}}} \right\} = \frac{1\frac{3}{4}}{\frac{1}{5}} \left. \vphantom{\frac{1\frac{3}{4}}{\frac{1}{5}}} \right\} = \frac{6\frac{3}{4}}{1} = \frac{27}{4}$$

$$\frac{7\frac{1}{2}}{3\frac{1}{4}} \left. \vphantom{\frac{7\frac{1}{2}}{3\frac{1}{4}}} \right\} = \frac{1\frac{5}{2}}{2\frac{1}{4}} \left. \vphantom{\frac{1\frac{5}{2}}{2\frac{1}{4}}} \right\} = \frac{1\frac{5}{2}}{1\frac{1}{2}} \left. \vphantom{\frac{1\frac{5}{2}}{1\frac{1}{2}}} \right\} = \frac{5\frac{5}{2}}{1\frac{1}{2}} = \frac{65 \times 2 \times 198}{4 \times 15 \times 35} = \frac{13 \times 33}{35}$$

$$\frac{5}{9} \left. \vphantom{\frac{5}{9}} \right\} = \frac{198}{9} = 22$$

$$\frac{27}{4} \div \frac{65 \times 33}{35} \div 22 = \frac{27 \times 35}{4 \times 65 \times 33} = \frac{13 \times 33}{35} = 12\frac{3}{35}$$

EXAMPLE 27.—Simplify

$$\frac{3\frac{1}{2}}{5} \div \frac{6}{3\frac{1}{4}} \div \frac{8\frac{1}{2}}{5} \div \frac{3}{3}$$

OPERATION.

$$\frac{3\frac{1}{2}}{5} \left. \vphantom{\frac{3\frac{1}{2}}{5}} \right\} = \frac{1\frac{3}{4}}{\frac{5}{1}} \left. \vphantom{\frac{1\frac{3}{4}}{\frac{5}{1}}} \right\} = \frac{13}{20}$$

$$\frac{6}{3\frac{1}{4}} \left. \vphantom{\frac{6}{3\frac{1}{4}}} \right\} = \frac{6}{1\frac{1}{4}} \left. \vphantom{\frac{6}{1\frac{1}{4}}} \right\} = \frac{24}{13}$$

$$\frac{8\frac{1}{2}}{5} \left. \vphantom{\frac{8\frac{1}{2}}{5}} \right\} = \frac{17}{10}$$

$$\frac{3}{3} = 1$$

$$\frac{13}{20} \div \frac{24}{13} \div \frac{17}{10} \div 1 = \frac{13 \times 13}{20 \times 24} \times \frac{10}{17} = \frac{13 \times 13}{20 \times 24} \times \frac{7 \times 5}{17}$$

$$= \frac{13 \times 13 \times 17}{20 \times 24 \times 7 \times 5} = \frac{2873}{16800} \text{ Ans.}$$

EXERCISE.

28. Multiply $\frac{12\frac{1}{2}}{7}$ by $\frac{\frac{2}{3} \text{ of } 32}{\frac{7}{5}}$ *Ans.* $2\frac{1}{4}$.
29. Divide $\frac{6\frac{1}{2}}{9\frac{1}{2}}$ by $\frac{\frac{6}{7}}{7}$ *Ans.* $1\frac{1}{5}$.
30. Divide $\frac{12\frac{1}{2}}{5\frac{1}{2}}$ by $\frac{2\frac{1}{2}}{5}$ *Ans.* $3\frac{1}{4}$.

51. From what has already been said, the truth of the following principles is evident.

1st. When the dividend is equal to the divisor, the quotient will be 1.

2nd. When the dividend is greater than the divisor, the quotient will be greater than 1.

3rd. When the dividend is less than the divisor, the quotient will be less than 1.

4th. The quotient will be as many times greater or less than 1 as the dividend is greater or less than the divisor.

5th. To divide a fraction by a whole number, we may either divide the numerator or multiply the denominator by that number.

6th. To divide a whole number by a fraction having 1 for its numerator, we simply multiply the whole number by the denominator of the fraction.

Thus, to divide by $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, &c., we multiply by 2, 3, 5, 7, &c.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numerals after the Questions refer to the numbered articles of the Section.

1. What is a fraction ? (1 and 3)
2. What does every fraction indicate ? (3)
3. What is the denominator of a fraction and why is it so called ? (4)
4. What is the numerator of a fraction and why is it so called ? (4)
5. What are the terms of a fraction ? (5)
6. How is the value of a fraction obtained ? (6)
7. When is the fraction equal to 1 and when greater or less than 1 ? (7)
8. What effect has multiplying the numerator of a fraction by any number ? (8)
9. How does multiplying the denominator of a fraction by any number affect the value of the fraction ? (8)
10. How does multiplying both terms of a fraction by the same number affect its value ? (8)
11. How does dividing the numerator by any number affect the value of the fraction ? (8)
12. How does dividing the denominator by any number affect the value of the fraction ? (8)
13. How does dividing both numerator and denominator by the same number affect the value ? (8)
14. Into what classes are fractions divided ? (9)
15. What is the distinction between vulgar and decimal fractions ? (10 and 11)
16. What is the meaning of the word "vulgar" as applied to fractions ? (11)
17. Enumerate the six different kinds of vulgar fractions.
18. What is a proper fraction ? (13)
19. What is an improper fraction ? (15)
20. What is a mixed number ? (16)
21. To what must an improper fraction always be equal ? (17)
22. What is a simple fraction ? (18)
23. What is a compound fraction ? (19)
24. What is a complex fraction ? (21)
25. How may we convert an integer into a fraction ? (23)
26. How may we reduce a whole number to a fraction having a given denominator ? (24)
27. How is a mixed number reduced to an improper fraction ? (25)
28. How is an improper fraction reduced to a mixed number ? (26)
29. How is a fraction reduced to its lowest terms ? (27 and 28)
30. How are fractions reduced to a common denominator ? (29)
31. How are fractions reduced to their least common denominator ? (30)
32. How is a compound fraction reduced to a simple one ? (31)

33. What is meant by cancelling ? (32)
34. Upon what principle may we cancel factors common to numerator and denominator ? (32 and 8)
35. How do we reduce complex fractions to simple ones ? (33)
36. What is a denominate fraction ? (34)
37. In what does reduction of denominate fractions consist ? (35)
38. How do we reduce a denominate fraction from a lower to a higher denomination ? (36)
39. How do we reduce a denominate fraction from a higher to a lower denomination ? (37)
40. How do we find the value of a denominate fraction ? (38)
41. How do we reduce one denominate number to the fraction of another ? (39)
42. What is addition of fractions ? (40)
43. What kind of fractions only can be added ? (41)
44. What is the rule for addition of fractions ? (42)
45. When mixed numbers are to be added how do we proceed ? (42 note)
46. What is the rule for the addition of denominate fractions ? (43)
47. What is the rule for subtraction of fractions ? (44)
48. What is the rule for multiplication of fractions ? (45)
49. Give a proof of the truth of this rule (45)
50. How do we multiply an integral denominate number by a fraction ? (46)
51. How may we multiply a fraction by a whole number ? (47)
52. How do we multiply a whole number by a fraction having 1 for numerator ? (47)
53. How do we multiply a whole number by a mixed number, the fractional part of which has 1 for numerator ? (47)
54. What is the rule for division of fractions ? (48)
55. Give a proof of the truth of this rule ? (48)
56. How do we divide an integral number by a fraction ? (49)
57. How do we divide a fraction by a whole number ? (51)
58. How do we divide a whole number by a fraction having 1 for its denominator ? (51)

MISCELLANEOUS EXERCISE ON VULGAR FRACTIONS.

1. The Ottawa River is 800 miles long ; the Gatineau 420 miles, the Chaudière 100 miles, the Richelieu 160 miles and the Niagara 35 miles. The entire length of the St. Lawrence, from the upper end of Lake Superior to the Sea is 2000 miles. How will the lengths of these different rivers be expressed as fractions of that of the St. Lawrence ?
2. The population of Goderich is $\frac{2}{3}$ of that of Peterborough, the population of Peterborough is $1\frac{1}{4}$ of that of Brockville, the population of Brockville is $1\frac{3}{8}$ of that of Prescott, the population of Prescott is $\frac{1}{2}$ of that of Ottawa City, the population of Ottawa City is $2\frac{1}{2}$ of that of Port Hope, and the population of Port Hope is $\frac{4}{5}$ of that of Toronto. What fraction is the population of Goderich of that of Toronto ?
3. What will $6\frac{1}{2}$ pounds of tea cost, at $65\frac{3}{4}$ cents per lb. ?
4. Suppose I have $\frac{1}{3}$ of a ship, and that I buy $\frac{1}{7}$ more; what is my entire share ?

5. A boy divided his marbles in the following manner; he gave to A $\frac{1}{2}$ of them, to B $\frac{1}{6}$, to C $\frac{1}{3}$, and to D $\frac{1}{6}$, keeping the rest to himself; how many did he give away, and how many did he keep?
6. Find the value of $\frac{5\frac{1}{2}-2\frac{1}{2}}{3\frac{3}{4}+2\frac{1}{2}}$ of $\frac{4\frac{1}{2}+5\frac{1}{2}\frac{2}{3}}{4\frac{1}{2}0}$ of $\frac{2\frac{3}{4}+1\frac{3}{4}}{7\frac{1}{2}\frac{3}{4}-2\frac{1}{4}}$
7. What cost 1670 $\frac{7}{8}$ pounds of coffee, at 12 $\frac{3}{4}$ cents per pound?
8. A tree, whose length was 136 feet, was broken into two pieces by falling; $\frac{2}{3}$ of the length of the longer piece equalled $\frac{1}{4}$ of the length of the shorter. What was the length of the two pieces respectively?
9. A farmer bought at one time 97 $\frac{1}{4}$ acres of land, for 1000 dollars, at another, 127 $\frac{2}{3}$ acres, for 1375 $\frac{1}{2}$ dollars; at another, 500 $\frac{2}{3}$ acres, for 6831 dollars; and at another, 333 $\frac{1}{2}$ acres, for 4013 $\frac{3}{15}$ dollars. What was the whole quantity of land that he purchased, and the sum that he paid for it?
10. Find the value of $(12\frac{2}{5}-8\frac{3}{4}-1\frac{1}{10}+1\frac{8}{10})\times 4\frac{1}{2}\times(7\frac{5}{12}-6\frac{1}{2})$, and also of $(\frac{2}{3}\div 1\frac{1}{2})-(\frac{5}{6}\div 3\frac{1}{11})$.
11. What is the value of 19 $\frac{1}{4}$ barrels of flour, at \$6 $\frac{3}{4}$ a barrel?
12. What is the value of 376 $\frac{1}{4}$ acres of land, at \$75 $\frac{2}{3}$ per acre?
13. Bought at one time 147 $\frac{2}{3}$ bushels of coal, and at another time 320 $\frac{1}{2}$ bushels. Having consumed 156 $\frac{1}{4}$ bushels, I desire to know what quantity of the coal purchased is still on hand?
14. Divide $\frac{7(\frac{1}{2}\text{ of } \frac{3}{4})}{\frac{1}{6}(\frac{3}{4}\text{ of } 7)}$ by 7 $\frac{7}{8}$; and find the value of $\frac{\frac{1}{2}+\frac{1}{3}+\frac{1}{4}}{\frac{1}{2\frac{1}{2}}+\frac{1}{3\frac{1}{4}}+\frac{1}{4\frac{1}{2}}}$
15. If 17 $\frac{1}{4}$ bushels of wheat sow 7 $\frac{1}{4}$ acres, how many bushels will it require to sow 1 acre?
16. Multiply the sum of 3 $\frac{3}{4}$, 4 $\frac{1}{4}$, and 4 $\frac{3}{8}$, by the difference of 7 $\frac{1}{2}$ and 5 $\frac{3}{8}$; and divide the product by the sum of 94 $\frac{1}{2}$ and 93 $\frac{1}{2}$.
17. Divide 2 by the sum of 2 $\frac{3}{4}$, $\frac{1}{8}$, and 4; add 1 $\frac{3}{4}$ - $\frac{7}{8}$ to the quotient; and multiply the result by the difference of 5 $\frac{1}{2}$ and 4 $\frac{1}{4}$.
18. Find the value of $(\frac{1}{2}+\frac{1}{3})\times(1\frac{1}{4}+2\frac{3}{4})\times(2\frac{1}{4}-1\frac{1}{2})\times(3\frac{1}{10}-\frac{7}{8})$; and also of $(1\frac{3}{4}\div 2\frac{1}{2})\times(5\frac{1}{2}\div 3\frac{1}{2})$.
19. A person dies worth \$40000, and leaves $\frac{1}{2}$ of his property to his wife, $\frac{1}{4}$ to his son, and the rest to his daughter. The wife at her death leaves $\frac{2}{3}$ of her legacy to the son, and the rest to her daughter; but the son adds his fortune to his sister's, and gives her $\frac{1}{2}$ of the whole. How much will the sister gain by this; and what fraction will her gain be of the whole?

DECIMALS AND DECIMAL FRACTIONS.

52. A decimal fraction is a fraction having unity with one or more 0s to the right of it for denominator:

Thus $\frac{4}{1000}$, $\frac{7}{100}$, $\frac{8}{10}$, $\frac{7}{100000}$, &c. are decimal fractions.

53. A decimal fraction is reduced to its corresponding *decimal* by dividing the numerator by the denominator, but since (Art. 51) this denominator is unity followed by one or more 0s, we divide the numerator by the denominator when we move the decimal point as many places to the left in the numerator as there are 0s in the denominator.

EXAMPLE 1. Reduce $\frac{743}{1000}$ to a decimal. *Ans.* .743.

2. Reduce $\frac{92376}{1000000}$ to a decimal. *Ans.* .00092376.

EXERCISE.

3. Reduce $\frac{567}{1000}$, $\frac{8}{100000}$ and $\frac{7}{10}$ to decimals. *Ans.* .567, .00098 and .7

4. Reduce $\frac{33}{100000}$ and $\frac{176}{1000000}$ to decimals. *Ans.* .000023 and .0000176.

5. Reduce $\frac{278643}{10000000}$ to a decimal. *Ans.* .000278643.

54. It is as inaccurate to confound a decimal fraction with its corresponding decimal as to confound a vulgar fraction with its quotient: Thus the value of $\frac{3}{4}$ is .75, so also the value of $\frac{75}{100}$ is .75 but .75 and $\frac{75}{100}$ are no more *identical* than are $\frac{3}{4}$ and .75.

55. To reduce a decimal to its corresponding decimal fraction:—

RULE.

Consider the significant part of the decimal as numerator and beneath it write for denominator 1 followed by as many 0s as there are places in the decimal.

EXAMPLE 6. Reduce .043 to a decimal fraction. *Ans.* $\frac{43}{1000}$.

7. Reduce .0000576 to a decimal fraction. *Ans.* $\frac{576}{10000000}$.

EXERCISE.

8. Reduce .73, .092 and .0003 to decimal fractions.

9. Reduce .137 and .00006943 to decimal fractions. *Ans.* $\frac{137}{1000}$, $\frac{82}{10000}$, and $\frac{73}{100000}$.

10. Reduce .13578967 and .023004003 to decimal fractions. *Ans.* $\frac{1357}{10000}$, and $\frac{2304003}{100000000}$.

Ans. $\frac{13578967}{100000000}$, and $\frac{23004003}{10000000000}$.

56. Decimal fractions follow exactly the same rules as vulgar fractions.—It is, however, generally more convenient to obtain their quotients, and then perform on them the required processes of addition, &c., by the methods already described (Sect. 11).

To reduce a vulgar fraction to a decimal or to a decimal fraction :—

RULE.

Divide the numerator by the denominator and the quotient will be the required "decimal"; the latter may be changed to its corresponding decimal fraction by (Art. 54).

This is merely actually performing the division which the fraction indicates.

EXAMPLE 11. Reduce $\frac{7}{8}$ to a decimal and also to a decimal fraction.

$$\begin{array}{r} 8 \overline{)7} \\ \underline{-875} \text{ Ans.} = \frac{875}{1000} \text{ Ans.} \end{array}$$

12. Reduce $\frac{9}{16}$ to a decimal.

$$\begin{array}{r} 16 \overline{)9} \\ \underline{-5625} \text{ Ans.} \end{array}$$

EXERCISE.

13. Reduce $\frac{1}{2}$ and $\frac{3}{4}$ to decimals. *Ans.* .5 and .75.
 14. Reduce $\frac{3}{8}$ and $\frac{1}{4}$ to decimal fractions. *Ans.* $\frac{375}{1000}$ and $\frac{250}{1000}$.
 15. Reduce $\frac{1}{2}$, $\frac{1}{12}$, and $\frac{1}{3}$ to decimals. *Ans.* .9733 +, .4666 + and .44117 +.*
 16. Reduce $\frac{7}{8}$, $\frac{1}{2}$, and $\frac{1}{3}$ to decimals. *Ans.* .857142 +, .4166 + and .44444 +.
 17. Reduce $\frac{1}{12}$ and $\frac{1}{1296}$ to decimals. *Ans.* .15178571428 + and .554012 +.

57. Let it be required to reduce £3 7s 6 $\frac{1}{2}$ d to the decimal of a pound.

OPERATION.

$\frac{1}{2}$ d = 7s d hence 6 $\frac{1}{2}$ d = 6.75d. If now we divide this by 12 we shall have its value as the decimal of a shilling.

6 $\frac{1}{2}$ d = 6.75d = .5625s hence 7s 6 $\frac{1}{2}$ d = 7.5625s.

Next if we divide this by 20 we shall have its value as a decimal of a pound.

7s. 6 $\frac{1}{2}$ d = 7.5625s = £.378125.

Therefore £3 7s 6 $\frac{1}{2}$ d = £3.378125.

Hence to reduce a denominate number of different denominations to an equivalent decimal of a given denomination we deduce the following :—

* The sign + written after these answers simply indicates that there is still a remainder and consequently that the division may be carried on further.

RULE.

Divide the lowest denomination named by that number which makes one of the next higher denomination.

Annex this quotient to the number of the next higher denomination given and divide as before.

Proceed thus through all the denominations to the one required, and the last result will be the one sought.

EXAMPLE 18. Reduce 3 days, 12 hours, 3 minutes, 30 seconds, to the decimal of a week.

OPERATION.

$$60)30 = \text{sec.} = 30 \text{ sec.}$$

$$60)3'5 = \text{decimal of a minute} = 3 \text{ min. } 30 \text{ sec.}$$

$$24)12'0583 = \text{decimal of an hour} = 12 \text{ h. } 3 \text{ m. } 30 \text{ sec.}$$

$$7)3'5024305 = \text{decimal of a day} = 3 \text{ days } 12 \text{ h. } 3 \text{ m. } 30 \text{ sec.}$$

$$\text{Ans. } ^\circ003472 = \text{decimal of a week} = 3 \text{ days } 12 \text{ h. } 3 \text{ m. } 30 \text{ sec.}$$

EXAMPLE 19. Reduce 187 lb. 13 oz. 11 drams to the decimal of a ton.

OPERATION.

$$60)11' \text{ drams.}$$

$$16)13'6875 \text{ ounces.}$$

$$2000)187'85546875 \text{ lbs.}$$

$$^\circ093927734375 \text{ ton. Ans.}$$

Here we divide the 11 drams by 16 and thus obtain $\cdot6875$ to which we prefix the given 13 oz. Next we divide this by 16 and obtain $\cdot85546875$ to which we bring down the 187 lb. and divide the result by 2000 the number of lbs. in a ton.

NOTE.—To divide by 2000 remove the decimal point *three* places to the left, and divide by 2; similarly to divide by 60, 20, &c., remove the decimal point *one* place to the left and divide by 6, 2, &c.

EXERCISE.

20. Reduce 3 yards 2 ft. 1 in. to decimal of a furlong.
Ans. $\cdot01679+$.
21. Reduce 3 dwt. 17 grs. Troy, to the decimal of a pound.
Ans. $\cdot01545138+$.
22. Reduce 2 scr. 7 grs. to the decimal of a pound, Apoth.
Ans. $\cdot0081597+$.
23. Reduce 5 fur. 35 per. 2 yd. 2 ft. 9 in. to the decimal of a mile.
Ans. $\cdot73603+$.
24. Reduce 3 qr. 2 na. to the decimal of a yard. *Ans.* $\cdot875$.
25. Reduce 5s. to the decimal of 13s. 4d. *Ans.* $\cdot375$.

* Reduce 5s. first to the fraction of 13s. 4d, and then reduce the resulting fraction to a decimal.

Thus 5s. reduced to the fraction of 13s. 4d. = $\frac{50}{130} = \frac{5}{13} = \cdot375$.

26. Reduce 12h. 55 min. 21 sec. to the decimal of a day.
Ans. .5384375.
27. Reduce $\frac{7}{8}$ of $\frac{1}{2}$ of 6½d. to the decimal of £½.
Ans. .012053+.
28. Reduce $\frac{2}{3}$ of $\frac{1}{2}$ of a mile to the decimal of 3½ inches.
Ans. 3620·571428+.
29. Reduce $\frac{1}{3}$ of $\frac{2}{3}$ of 3½ lb. Avoir. to the decimal of $\frac{1}{4}$ of an oz.
Ans. 9·2444+.
30. Reduce 3 pk. 1 gal. 1 qt. 1 pt. to the decimal of a bushel.
Ans. .921875.

58. Let it be required to find the value in terms of a lower denomination of .7825 of a yard.

OPERATION.

```

  7825
   3
  ---
23475
  12
  ---
41700
  12
  ---
20400

```

Ans. 2 ft. 4 in. 2·04 lines.

EXPLANATION.—Since there are 3 feet in a yard, it is evident that any decimal of a yard is *three* times as great a decimal of a foot. Hence, to reduce the decimal of a yard to a decimal of a foot we multiply it by 3. This gives us two feet and .3475 of a foot. Similarly multiplying the decimal of a foot by 12 reduces it to an equivalent decimal of an inch. We thus find .3475 of a foot equal to 4 inches and .17 of an inch. Again, multiplying this last by 12 reduces it to the decimal of a line, and we thus find the whole quantity of .6825 of a yard equal to 2 ft. 4 in. 2·04 lines.

NOTE.—In these multiplications we only multiply the number to the right of the separating point.

Hence, to find the value of a denominate number in terms of integers of a lower denomination we have the following:—

RULE.

Multiply the given decimal by the number of units of the next lower denomination that make one of the given denomination.

Point off as many decimal places as there were in the multiplier, and the integral portion, if any, will be units of that lower denomination; the decimal part may be reduced to a still lower denomination and so on.

EXAMPLE 31.—Find the value of £·97875.

OPERATION.

```

  97875
   20
  ---
1957500s.
   12
  ---
690000d.
   4
  ---
360000f.

```

Ans. 19s. 6½d. + $\frac{2}{3}$ of a farthing.

EXAMPLE 32.—Find the value of $\cdot 7863625$ of a pound Apothecaries' weight.

OPERATION.

$$\begin{array}{r} \cdot 7863625 \\ \underline{12} \\ 9'4383500 \text{ oz.} \\ \underline{8} \\ 3'4908000 \text{ drs.} \\ \underline{3} \\ 1'4724000 \text{ scr} \\ \underline{20} \\ 9'4480000 \text{ grs.} \end{array}$$

Ans. 9 oz. 3 dr. 1 scr. 9'448 grains.

EXERCISE.

33. Find the value of $0\cdot 3945$ of a day.
Ans. 9 hours 28 min. 4·8 sec.
34. Find the value of $0\cdot 3965$ of a mile.
Ans. 3 fur. 6 per. 4 yds. 2 ft. 6'24 in.
35. Find the value of $0\cdot 309153$ of an oz. Troy.
Ans. 6 dwt. 4'39344 grains.
36. Find the value of $22\cdot 75$ of £2 2s. 6d. *Ans.* £48 6s. 10jd.
37. Find the value of $11\cdot 17825$ of 7 bush. 1 pk. 1 gal. 1 qt.
*Ans.** 82 bush. 3 pks. 0 gal. 1 qt. 0'4905 pt.
38. Find the value of $\cdot 2057$ of a lb. Troy.
Ans. 2 oz. 9 dwt. 8'832 grains.
39. Find the value of $\cdot 176$ of 1 fur. 36 per. 2 yds. 5 in.
Ans. 13 per. 2 yds. 1 ft. 4 in.
40. Find the value of $\cdot 625$ of a league. *Ans.* 1 mile 7 fur.
41. What is the value of $\cdot 015625$ of a bushel? *Ans.* 1 pint.
42. What is the value of $\cdot 9378$ of an acre?
Ans. 3 roods 30 per. 1 yd. 4 ft. $9\frac{9}{25}$ inches.
43. Find the value of $\cdot 2775$ of 1 sq. yd. 3 ft. 72 in.
Ans. 3 sq. ft. $67\frac{1}{2}$ in.

CIRCULATING OR REPEATING DECIMALS.

59. Let it be required to reduce $\frac{6}{7}$ and $\frac{5}{7}$ to decimals.

OPERATION.

$$\begin{array}{r} 9)5 \\ \underline{55}555 \text{ \&c.} \\ 7)6 \\ \underline{57}142857142857142 \text{ \&c.} \end{array}$$

* If the given quantity be expressed in more than one denomination it should be reduced to *one* before applying the rule. Thus in this example 7 bush. 1 pk. 1 gal. 1 qt. = 237 qrts. and $11\cdot 17825 \times 237 = 2648'24525$ qts. = 82 bush. 3 pks. 0 gal. 1 qt. 0'4905 pints.

In these and many other cases the division does not terminate, and the value of the fraction can only be approximately expressed. In the former of the above examples the figure 5 is constantly repeated, and in the latter the series of figures 857142.

60. Decimals which do not terminate, *i. e.*, which consist of the same digit or set of digits constantly repeated, are called Repeating or Circulating Decimals.

61. The digit or set of digits, which repeats, is called a *repetend*, *period* or *circle*.

NOTE.—The terms *period* and *circle* are commonly used only when the repetend contains two or more digits.

62. A Single repetend is one in which only a single digit repeats,

Thus, $\cdot 3333$ &c. ; $\cdot 7777$ &c. ; $\cdot 8888$ &c. are single repetends.

63. A Single repetend is expressed by writing the digit that repeats with a dot over it,

Thus, $\cdot 333$ &c. is written $\cdot \dot{3}$; $\cdot 777$ &c. is written $\cdot \dot{7}$.

64. A Circulating Decimal or Compound Repetend is one in which more than one digit repeats,

Thus, $\cdot 347347347$ &c. ; $\cdot 202020$ &c. ; $\cdot 123412341234$ &c. are Circulating Decimals or Compound Repetends.

65. A Circulating Decimal is expressed by writing the recurring period once with a dot over its first and last digits.

Thus, $\cdot 347347$ &c. is written $\cdot \dot{3}47$; $\cdot 2020$ &c. is written $\cdot \dot{2}0$; $\cdot 12341234$ &c. is written $\cdot \dot{1}234$.

66. A Pure Repetend or Circulating Decimal is one in which the repetend commences *immediately* after the decimal point.

67. A Mixed Repetend or Circulating Decimal is one which contains one or more ciphers or significant figures between the repetend and the decimal point,

Thus, $\cdot 3$, $\cdot 7$, $\cdot 1$ are Pure Repetends.

$\cdot 78917$, $\cdot 0378$, $\cdot 002$ are Mixed Repetends.

$\cdot \dot{7}2$, $\cdot 0\dot{4}3$, $\cdot 8137\dot{8}$ are Pure Circulating Decimals.

$\cdot 137\dot{8}$, $\cdot 07820\dot{5}$, $\cdot 071786\dot{8}$ are Mixed Circulating Decimals.

68. Similar Repetends are those which commence at the same number of places from the decimal point,

Thus, $\cdot 7134\dot{5}$, $\cdot 91278\dot{6}$ and $\cdot 0007134\dot{6}$ are Similar Repetends.

69. Dissimilar Repetends are those which commence at a different number of places from the decimal point,

Thus, $\cdot 734\dot{2}$, $\cdot 92862\dot{7}$ and $\cdot 913427\dot{8}$ are Dissimilar Repetends.

70. Coterminous Repetends are those which terminate at the same number of places from the decimal point,

Thus, $\cdot 7437\dot{6}24\dot{3}$ and $\cdot 134\dot{7}$ are Coterminous Repetends.

71. Similar and Coterminous Repetends are those which both commence and end at the same distance from the decimal point,

Thus, $\cdot 7342\dot{6}7$, $16\cdot 47121\dot{2}$, $198\cdot 16134\dot{1}$ are Similar & Coterminous Repetends.

72. In reducing a fraction to a decimal we place a point after the numerator, and annex 0s to it until it is exactly divisible by the denominator. But since the point does not affect the division, merely determining the *place* of the point in the resulting quotient, it is manifest that we may leave it altogether out of consideration, so that annexing 0s to the numerator becomes in effect *multiplying it by such a power of 10 as will make it contain the denominator*. Now if the fraction, before proceeding to the division, be reduced to its lowest terms, the denominator can have no factor in common with the numerator; and if the denominator be exactly contained in the numerator with the 0s annexed, it can only be from its being contained in that power of 10 by which the original numerator was multiplied. But since 10 contains only the factors 2 and 5, any power of 10 can contain only the factors 2 and 5; and hence, in order that the denominator may be exactly contained in the numerator with 0s annexed, it must contain only the factors 2 and 5, or powers of 2 and 5.

Hence, when a vulgar fraction is reduced to its lowest terms, if the denominator contain no factors other than 2 and 5, the corresponding decimal will be *finite*, but if the denominator contain any other factor than 2 and 5, as 3, 7, 11, &c., the corresponding decimal will be *infinite*, i. e., will be a repetend.

EXAMPLE 44.—Can $\frac{1}{6}$, $\frac{1}{12}$, $\frac{1}{15}$ and $\frac{1}{18}$ be exactly expressed as decimals?

16, the denominator of the first, $= 2 \times 2 \times 2 \times 2$, (i. e. contains no prime factor other than 2 or 5) therefore it *can* be exactly expressed by a decimal.

25 $= 5 \times 5$ (i. e. no prime factor other than 2 or 5) therefore $\frac{1}{12}$ *can* be exactly expressed by a decimal.

12 $= 2 \times 2 \times 3$ (i. e. *does* contain a factor other than 2 or 5) therefore $\frac{1}{15}$ *cannot* be exactly decimated.

125 $= 5 \times 5 \times 5$ (i. e. no factor other than 2 or 5) therefore $\frac{1}{18}$ *can* be exactly decimated.

EXERCISE.

Of the following fractions, which can and which cannot be exactly decimated, i. e., reduced to equivalent decimals?

45. $\frac{7}{8}$, $\frac{17}{25}$, $\frac{13}{32}$, $\frac{1}{1024}$, and $\frac{1}{303}$.

46. $\frac{6}{75}$, $\frac{1}{8}$, $\frac{2}{27}$, $\frac{5}{100}$, $\frac{11}{64}$.

47. $\frac{1}{24}$, $\frac{1}{11}$, $\frac{1}{5}$, $\frac{1}{3}$, and $\frac{1}{1280}$.

73. We may determine the number of places in the decimal or finite part of the decimal corresponding to a vulgar fraction by the following:—

RULE.

Reduce the fraction to its lowest terms, and decompose the denominator into its prime factors.

If the denominator contains no factors other than 2 or 5, or powers of 2 or 5 the whole decimal is finite.

If the denominator does not contain 2 or 5 as factor, the decimal contains no finite part.

The highest exponent of 2 or 5 will indicate the number of decimal places in the finite part of the corresponding decimal.

EXAMPLE 48.—How many decimal places will be required to express $\frac{117}{3125}$?

Here, $3125 = 5 \times 5 \times 5 \times 5 \times 5 = 5^5$. Therefore the equivalent decimal will contain 5 places.

EXAMPLE 49.—How many decimal places will be required to express $\frac{1600}{16000}$?

Here, $1600 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^6 \times 5^2$. Hence 6 is the highest exponent, and the number of decimal places will therefore be 6.

EXERCISE.

50. How many decimal places will be required to express the following fractions, viz:— $\frac{1}{6}$, $\frac{9}{25}$, $\frac{111}{8000}$, and $\frac{133}{1024}$?

Ans. 4, 3, 6 and 10.

51. How many places will there be in the finite part of the decimals corresponding to $\frac{7}{25}$, $\frac{111}{256}$, $\frac{133}{1024}$, and $\frac{133}{64}$?

Ans. 5, 7, 4 and 11.

74. In decimating vulgar fractions where many places are required in the decimal, the method of continually dividing becomes very tedious. In such cases we may sometimes shorten the work as follows:—

EXAMPLE 52.—What decimal is equivalent to the vulgar fraction $\frac{1}{27}$?

| | |
|----------------|--|
| OPERATION. | |
| 29)100(0'03448 | |
| 87 | |
| 130 | |
| 116 | |
| 140 | |
| 116 | |
| 240 | |
| 233 | |
| 8 | |

$\frac{1}{9} = 0.03448\overline{27}$. Therefore $\frac{8}{9} = 0.27586\overline{27}$, and substituting this value for $\frac{8}{9}$ we get:—

$\frac{1}{9} = 0.0344827586\overline{27}$. Hence $\frac{6}{9} = 0.2068965517\overline{27}$ and substituting this for $\frac{6}{9}$ we get:—

$\frac{1}{9} = 0.03448275862068965517\overline{27}$. Hence $\frac{7}{9} = 0.241379310344-82758620\overline{27}$ and substituting this value for $\frac{7}{9}$ we get:—

$\frac{1}{9} = 0.0344827586206896551724137931. \textit{Ans.}$

75. The number of places in a period cannot exceed the units in the denominator minus one.

This is manifest from the fact that all the remainders that occur must be less than the denominator, and their number cannot be greater than the denominator, minus one; because we carry on the division by affixing 0s, and it follows that whenever we obtain a remainder like one that has previously occurred, the digits of the decimal will begin to repeat,

Thus $\frac{5}{7} = 0.714\overline{285714}$, where the small figures above the line represent the successive remainders, none of which, of course, can be as great as 7, the divisor,—the next remainder after the 6 would be 4 and consequently the digits would commence to repeat.

76. Those repetends that have as many places, minus one, as there are units in the denominators of their equivalent vulgar fractions are sometimes called *perfect repetends*.

The following are the only fractions having a denominator less than 100 that give *perfect repetends* when decimated:—

$$\frac{1}{7}, \frac{1}{17}, \frac{1}{19}, \frac{1}{23}, \frac{1}{29}, \frac{1}{47}, \frac{1}{53}, \frac{1}{71}, \text{ and } \frac{1}{97}.$$

77. To reduce a pure repetend to an equivalent vulgar fraction:—

RULE.

Put the period for numerator and as many nines as there are places in the period for denominator.

EXAMPLE 53. What vulgar fractions are equivalent to $\cdot\dot{7}$, $\cdot\dot{93}$, $\cdot\dot{704}$ and $\cdot\dot{007043}$.

Ans. $\cdot\dot{7} = \frac{7}{9}$; $\cdot\dot{93} = \frac{93}{99} = \frac{31}{33}$; $\cdot\dot{704} = \frac{704}{999}$; $\cdot\dot{007043} = \frac{7043}{999999}$.

Reason $\frac{1}{9} = \cdot\dot{1}$ therefore $\frac{2}{9}$, $\frac{3}{9}$, $\frac{4}{9}$, &c., = $\cdot\dot{2}$, $\cdot\dot{3}$, $\cdot\dot{4}$, &c., hence $\cdot\dot{1}$, $\cdot\dot{2}$, $\cdot\dot{3}$, &c., = $\frac{1}{9}$, $\frac{2}{9}$, $\frac{3}{9}$, &c.

Similarly $\frac{1}{99} = \cdot\dot{01}$, therefore $\frac{7}{99} = \cdot\dot{07}$; $\frac{93}{99} = \cdot\dot{23}$; $\frac{704}{999} = \cdot\dot{70}$; &c.

Hence $\cdot\dot{01} = \frac{1}{99}$; $\cdot\dot{07} = \frac{7}{99}$; $\cdot\dot{23} = \frac{23}{99}$; $\cdot\dot{17} = \frac{17}{99}$; &c.

So also $\frac{1}{999} = \cdot\dot{001}$; $\frac{704}{999} = \cdot\dot{005}$; $\frac{7043}{9999} = \cdot\dot{167}$; &c.

Hence $\cdot\dot{001} = \frac{1}{999}$; $\cdot\dot{243} = \frac{243}{999}$; &c., whence the reason of the rule is evident.

EXERCISE.

54. Reduce $\cdot\dot{8}$, $\cdot\dot{06}$, $\cdot\dot{342}$, $\cdot\dot{7004}$ and $\cdot\dot{002003}$ to equivalent vulgar fractions.

Ans. $\frac{8}{9}$, $\frac{6}{99}$, $\frac{342}{999} = \frac{38}{111}$, $\frac{7004}{9999}$ and $\frac{2003}{999999}$.

55. Reduce $\cdot\dot{19}$, $\cdot\dot{1067}$, $\cdot\dot{11115}$ and $\cdot\dot{704103}$ to equivalent vulgar fractions.

Ans. $\frac{19}{99}$, $\frac{1067}{9999} = \frac{97}{909}$, $\frac{11115}{99999} = \frac{11115}{111111}$ and $\frac{704103}{999999} = \frac{704103}{999999}$.

56. Reduce $\cdot\dot{102}$, $\cdot\dot{0013}$, $\cdot\dot{00007103}$, $\cdot\dot{01020304}$ and $\cdot\dot{987654321}$ to equivalent vulgar fractions.

Ans. $\frac{102}{999}$, $\frac{13}{9999}$, $\frac{7103}{99999999}$, $\frac{1020304}{99999999}$ and $\frac{987654321}{1111111111}$.

78. To reduce a mixed repetend to an equivalent vulgar fraction:—

RULE.

Subtract the finite part from the whole and set down the difference for the numerator.

For denominator put as many 9s as there are places in the 'infinite' part followed by as many 0s as there are places in the 'finite' part.

EXAMPLE 57. Reduce $\cdot\dot{73}$, $\cdot\dot{1234}$ and $\cdot\dot{7132092}$ to their equivalent vulgar fractions.

OPERATION.

$73 - 7 = 66 =$ numerator of first fraction.
 $1234 - 12 = 1222 =$ " second "
 $7132092 - 713 = 7131379 =$ " third "
 $90 =$ 1st Denominator, since the repetend contains one place in the finite, and one place in the infinite part.
 $9900 =$ 2nd Denominator, since the repetend contains two places in the finite part and two in the infinite part.

999900 = 3rd Denominator, since the infinite part of the decimal, contains *four* places and the finite part *three* places.

Hence, $\cdot 73 = \frac{73}{100} = \frac{1}{10} + \frac{1}{100}$, $\cdot 1234 = \frac{1234}{10000} = \frac{1}{1000} + \frac{1}{10000}$ and $\cdot 7132092 = \frac{7132092}{10000000}$.

REASON.—Let it be required to reduce $\cdot 978734$ to an equivalent vulgar fraction.

$$\text{Let } x = \cdot 978734 \quad (\text{I})$$

$$\text{Then } 100x = 97\cdot 8734 \quad (\text{II})$$

And $1000000x = 978734\cdot 8734$ (III); subtracting (II) from (III) gives $999900x = 978734 - 97$

Whence $x = \frac{978734 - 97}{999900} =$ *Whole repetend minus the finite part for numerator; and as many 9s as there are places in infinite part, followed by as many 0s as there are places in finite part for denominator.*

The rule may also be explained as follows:—

Taking the same example $\cdot 978734$ and multiplying it by 100, we get $\cdot 978734 \times 100 = 97\cdot 8734 = 97 + \cdot 8734 = 97 + \frac{8734}{10000}$ (Art. 77.)

Now, since we multiplied by 100 this result is 100 times too great. Therefore $\cdot 978734 = \frac{97}{100} + \frac{8734}{999900}$, and to add these fractions we must reduce them to a common denominator when they become:—

$$\frac{97 \times (10000 - 1)}{999900} + \frac{8734}{999900} = \frac{97 \times 10000 - 97}{999900} + \frac{8734}{999900} = \frac{970000 - 97}{999900} + \frac{8734}{999900}$$

$= \frac{978734 - 97}{999900} =$ *Whole repetend minus finite part for numerator; and as many 9s as there are places in infinite part, followed by as many 0s as there are places in finite part, for denominator.*

Whence the truth of the rule is manifest.

EXERCISE.

58. Reduce $\cdot 8325$, $\cdot 147658$, and $\cdot 4320075$ to their equivalent vulgar fractions.

Ans. $\frac{8325}{10000} = \frac{1177}{1250}$, $\frac{147658}{1000000} = \frac{147658}{1000000}$, and $\frac{4320075}{10000000} = \frac{172803}{400000}$.

59. Reduce $875\cdot 4965$ and $301\cdot 82756$ to their equivalent mixed numbers.

Ans. $875\frac{1241}{2500}$ and $301\frac{1111}{1250}$.

60. Reduce $\cdot 083$, $\cdot 0714285$, and $\cdot 123456$ to their equivalent vulgar fractions.

Ans. $\frac{1}{12}$, $\frac{1}{14}$, and $\frac{1111}{9000}$.

61. Reduce $\cdot 7034$, $\cdot 96432$, $\cdot 00207$, and $\cdot 143271$ to their equivalent vulgar fractions.

Ans. $\frac{3517}{5000}$, $\frac{24144}{2500}$, $\frac{1}{5000}$, and $\frac{11111}{77000}$.

79. There are several properties belonging to repetends which it is necessary to remember. They are as follows :

1st. Any finite decimal may be regarded as a repetend if we make the 0s recur :

Thus, $\cdot 27 = \cdot 270 = \cdot 2700 = \cdot 27000 = \cdot 270000, \&c.$

2nd. A repetend having any number of places may be reduced to one having *twice, thrice, &c.*, that number of places.

Thus a repetend having 2 places may be reduced to one having 4, 6, 8, 10, 12, &c., places.

For example, $\cdot 37\bar{2} = \cdot 3727\bar{2} = \cdot 372727\bar{2}, \&c.$

$\cdot 23\bar{2}134 = \cdot 2321342134 = \cdot 23213421342134, \&c.$

3rd. Two or more repetends, having a different number of places in each, may be reduced to others having the same number of places in each, by the following :—

RULE.

Take the numbers indicating how many places there are in each repetend, and find their least common multiple. Reduce each repetend to that number of places.

Thus, let it be required to reduce $\cdot 14\bar{7}$, $\cdot 93\bar{2}$, $\cdot 841\bar{7}$, to repetends having the same number of places.

Here the numbers of places are 1, 2 and 3, and the least common multiple of 1, 2 and 3 is 6, and hence each new repetend must have 6 places.

Therefore, $\cdot 14\bar{7} = \cdot 147777\bar{7}$, $\cdot 93\bar{2} = \cdot 932323\bar{2}$, and $\cdot 841\bar{7} = \cdot 841741\bar{7}$.

4th. Any repetend may be transformed into another having a finite part and an infinite part containing as many places as the original repetend, and hence any two or more repetends may be made similar,

Thus, $\cdot 412\bar{3} = \cdot 41231\bar{2} = \cdot 41231\bar{2}, \&c.$

$7\cdot 654321 = 7\cdot 6543216 = 7\cdot 65432165, \&c.$

5th. Having made two or more repetends similar by the last article, they may be made coterminous by the preceding one, and hence two or more repetends may always be made *similar* and *coterminous*.

6th. If several repetends of equal places be added together their sum will be a repetend of the same number of places; since every set of periods will give the same sum.

ADDITION OF CIRCULATING DECIMALS.

To add circulating decimals:—

RULE.

80. *Make the repetends similar and coterminous and write them under one another, so as to have the units of the same order in the same vertical column.*

Add, beginning at the right hand side and carrying what would have been obtained if the decimals had been carried out two or three places further.

EXAMPLE 62.—Add together $\cdot 78\dot{3}$, $\cdot 92\dot{7}$, $\cdot 42\dot{1}$ and $9\cdot 12345\dot{6}$.

| Dissimilar. | = | Similar. | = | Similar and Coterminous. |
|-----------------------|---|-----------------------|---|-------------------------------|
| $\cdot 78\dot{3}$ | = | $\cdot 78\dot{3}$ | = | $\cdot 783333333333\dot{3}$ |
| $\cdot 92\dot{7}$ | = | $\cdot 927\dot{2}$ | = | $\cdot 927272727272\dot{2}$ |
| $\cdot 42\dot{1}$ | = | $\cdot 4214\dot{2}$ | = | $\cdot 4214214214214\dot{2}$ |
| $9\cdot 12345\dot{6}$ | = | $9\cdot 12345\dot{6}$ | = | $9\cdot 1234563456345\dot{6}$ |
| | | | | 1 carried. |

Sum, = $11\cdot 2554838276620\dot{4}$

EXERCISE.

63. Add together $\cdot 9$, $6\cdot 32\dot{7}$, $19\cdot 43$, $27\cdot 027\dot{8}$ and $\cdot 034712\dot{3}$.

Ans. $53\cdot 819863827\dot{4}$.

64. Add together $7\cdot 42\dot{7}$, $9\cdot 123\dot{4}$, $17\cdot 298764\dot{3}$ and $18\cdot 6\dot{7}$.

Ans. $52\cdot 52622820390147\dot{1}$.

65. Add together $4\cdot 9\dot{5}$, $7\cdot 16\dot{4}$, $4\cdot 712\dot{3}$ and $\cdot 9731\dot{7}$.

Ans. $17\cdot 809250213\dot{8}$.

66. Add together $1\cdot 5$, $99\cdot 083$, $\cdot 16\dot{2}$, $\cdot 81\dot{4}$, $2\cdot 9\dot{3}$, $3\cdot 769230$, $97\cdot 26$
and $134\cdot 0\dot{9}$.

Ans. $339\cdot 62617744\dot{3}$.

SUBTRACTION OF CIRCULATING DECIMALS.

To subtract one repetend from another:—

RULE.

Make the repetends similar and coterminous, and write one be

neath the other, so as to have units of the same order in the same vertical column.

Subtract as in whole numbers, taking notice whether one would have been borrowed if the periods had been extended.

EXAMPLE 67.—From $97\cdot0342\dot{9}$ take $11\cdot038\dot{7}6$.

| Dissimilar. | Similar. | Similar and Coterminous. |
|----------------------|-----------------------|--------------------------|
| $97\cdot0342\dot{9}$ | $97\cdot0342\dot{9}$ | $97\cdot0342929\dot{2}9$ |
| $11\cdot038\dot{7}6$ | $11\cdot038\dot{7}68$ | $11\cdot038768\dot{8}$ |

True difference, $85\cdot99552416\dot{0}$

If the periods had been extended we would have had to borrow one from the last figure of the minuend period, and bearing this in mind we say 8 from 8, 0, &c.

EXERCISE.

68. From $729\cdot342\dot{7}$ take $93\cdot126$. *Ans.* $636\cdot216\dot{7}42$.
69. From $1\cdot43\dot{7}29\dot{1}$ take $\cdot007\dot{1}3$. *Ans.* $1\cdot4301600597824$.
70. From $1\cdot127\dot{5}4$ take $\cdot4738\dot{4}$. *Ans.* $\cdot6537001628090\dot{7}$.
71. From $42\cdot1876\dot{3}$ take $17\cdot000000843\dot{2}$. *Ans.* $25\cdot187632490\dot{0}$.

MULTIPLICATION OF CIRCULATING DECIMALS.

81. To multiply one repetend by another or by a finite decimal :—

RULE.

Change the decimals into their equivalent vulgar fractions (*Arts. 76 and 77*), multiply these together, and reduce the product to its equivalent decimal.

EXAMPLE 72.—Multiply $\cdot\dot{3}$ by $\cdot\dot{7}8$.

$$\cdot\dot{3} = \frac{3}{9} = \frac{1}{3} \text{ and } \cdot\dot{7}8 = \frac{78}{99} = \frac{26}{33}.$$

Therefore, $\cdot\dot{3} \times \cdot\dot{7}8 = \frac{1}{3} \times \frac{26}{33} = \frac{26}{99} = 26\dot{2}6$ *Ans.*

EXAMPLE 73.—Multiply $\cdot\dot{3}18$ by $\cdot\dot{7}432$.

$$\cdot\dot{3}18 = \frac{318}{999} \text{ and } \cdot\dot{7}432 = \frac{7432}{9999}.$$

Therefore $\cdot\dot{3}18 \times \cdot\dot{7}432 = \frac{318}{999} \times \frac{7432}{9999} = \frac{23648}{19998} = 23648\dot{}$.

EXERCISE.

74. Multiply $7\cdot25$ by $2\cdot9$. *Ans.* $21\cdot75$.
75. Multiply $\cdot29\dot{7}$ by $7\cdot72$. *Ans.* $2\cdot2951\dot{3}$.
76. Multiply $\cdot81\dot{8}$ by $\cdot77$. *Ans.* $\cdot63$.
77. Multiply $1\cdot73\dot{5}$ by $\cdot4705\dot{3}$. *Ans.* $\cdot81654168350$.
78. Multiply $4\cdot72\dot{2}$ by $\cdot19\dot{8}$. *Ans.* $\cdot93\dot{5}$.

DIVISION OF CIRCULATING DECIMALS.

82. To divide one repetend by another or by a finite decimal:—

RULE.

Change the decimals into their equivalent vulgar fractions, divide as in Art. 48, and reduce the result to its corresponding decimal.

EXAMPLE 79.—Divide $\cdot42\dot{7}$ by $\cdot81\dot{8}$.

$$\cdot42\dot{7} = \frac{427}{1000} \text{ and } \cdot81\dot{8} = \frac{818}{1000}$$

Therefore, $\cdot42\dot{7} \div \cdot81\dot{8} = \frac{427}{1000} \div \frac{818}{1000} = \frac{427}{818} = \frac{427}{1000} \times \frac{1000}{818} = \frac{427}{818} = 0\cdot52$.

EXERCISE.

80. Divide $\cdot082$ by $\cdot12\dot{3}$. *Ans.* $\cdot6$.
81. Divide $389\cdot18\dot{5}$ by $15\cdot7$. *Ans.* $24\cdot6$.
82. Divide $\cdot81654168350$ by $\cdot4705\dot{3}$. *Ans.* $1\cdot73\dot{5}$.
83. Divide $\cdot4\dot{5}$ by $\cdot118881$. *Ans.* $3\cdot823529411764705\dot{8}$.

MISCELLANEOUS EXERCISE ON DECIMALS.

84. Reduce $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{4}{5}$ of 14 to its equivalent decimal.
85. Multiply $\cdot67$ by $2\cdot1\dot{3}$.
86. Find the value of $\cdot678125$ of a week.
87. Reduce $\cdot9243\dot{7}$ to its equivalent fraction.
88. Add together $67\cdot234,98\cdot71\dot{3}$, and $91\cdot0347123\dot{4}$, and from their sum take $100\cdot12345678\dot{9}$.
89. Reduce 5 fur. 36 rds. 2 yds. 2 ft. 9 in. to the decimal of a mile.

90. Find the difference between $17\dot{4}2857\dot{1}$ sq. ft. and $100\dot{8}$ sq. in.
91. What is the value of $\cdot 917897\dot{7}\dot{2}$ of two acres ?
92. Reduce $11\cdot 28\dot{7}$ and $1\cdot 04\dot{2}857\dot{1}$ to vulgar fractions.
93. Divide $47\cdot 345$ by $1\cdot 7\dot{6}$.
94. From $85\cdot 6\dot{2}$ take $13\cdot 7643\dot{2}$.
95. What is the difference between $\cdot 734$ of a lb. and $\cdot 198$ of an oz. avoirdupois ?
96. How many yards of carpet 2 ft. $5\frac{1}{2}$ in. wide will be required to cover a floor $27\cdot 3$ ft. long and $20\cdot 16$ ft. wide.
97. Multiply $3\cdot 14\dot{5}$ by $4\cdot 29\dot{7}$.
98. How many finite places are there in the decimals corresponding to $\frac{3}{10}$, $\frac{7}{4}$, $\frac{8}{15}$, $\frac{11}{44}$, $\frac{9}{60}$, and $\frac{11}{33\frac{1}{4}}$?
99. Add together $81\frac{1}{3}$, $61\cdot 12\dot{6}$, $328\frac{2}{3}$, and $5\cdot 62\dot{4}$.
100. Reduce $\left(\frac{4\cdot 4 - 2\cdot 83}{1\cdot 6 + 2\cdot 629}\right)$ of $\frac{6\cdot 8 \text{ of } 3}{2\cdot 25}$ + $\frac{2\cdot 8 \text{ of } 2\cdot 27}{1\cdot 13\dot{6}}$ to a simple quantity.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers after the questions refer to the articles of the Section.

1. What is a decimal fraction ? (52)
2. What is the distinction between a decimal and its corresponding decimal fraction ? (54 and Art. 47 Sect. I.)
3. How is a decimal reduced to its corresponding decimal fraction ? (55)
4. How is a vulgar fraction reduced to a decimal ? (56)
5. How would you reduce 4 oz. 17 dwt. 16 grs. to the decimal of a lb ? (57)
6. How would you find the value of $\cdot 71345$ of a French ell ? (58)
7. What is meant by repeating or circulating decimals ? (60)
8. What is a repetend, period, or circle ? (61)
9. What is a single repetend, and how is it expressed ? (62 & 63)
10. What is a circulating decimal or compound repetend, and how is it expressed ? (64 & 65)
11. What is a pure repetend ? (66)
12. What is a mixed repetend ? (67)
13. What are similar repetends ? Give an example. (68)
14. What are dissimilar repetends ? Give examples. (69)
15. What are coterminous repetends ? Give examples. (70)
16. When are repetends said to be both similar and coterminous ? Give examples. (71)
17. When can a vulgar fraction be exactly expressed by a decimal ? (72)
18. Show that this must necessarily be the case. (72)
19. How can we ascertain the number of places in the finite part of the decimal corresponding to any vulgar fraction ? (73)
20. If the decimal corresponding to any vulgar fraction contain a repetend, what is the greatest number of places that repetend can contain ? (75)
21. Show that this must necessarily be the case.
22. What are perfect repetends ? (76)
23. How is a pure repetend reduced to a vulgar fraction ? (77)

24. How is a mixed repetend reduced to a vulgar fraction ? (78)
25. Show the truth of this rule. (77)
26. Show that any finite decimal may be made into a repetend. (79)
27. Show that any repetend may be reduced to another having twice, thrice, &c., as many places. (79)
28. Show that any number of repetends may be made to have the same number of places, and give the rule. (79)
29. Show that any pure repetend may be transformed into a mixed repetend. (79)
30. Show that two or more repetends may be made similar and coterminous. (79)
31. How are circulating decimals added ? (80)
32. How are circulating decimals subtracted ? (81)
33. How do we multiply circulating decimals together ? (82)
34. How do we divide one circulating decimal by another ? (83)

MISCELLANEOUS EXERCISE.

(On preceding Rules.)

1. Transform 4312131 *quinary*, into the *nonary*, *ternary*, and *octenary* scales, and prove the results by reducing all four numbers to the decimal scale.
2. Write down seven hundred and two trillions seven millions thirty thousand and seventeen, and four millions and seventy-six tenths of quadrillionths.
3. Divide $976\cdot432$ by $\cdot00000096$.
4. What is the value of
$$\frac{(2\frac{1}{2} + \cdot5625 - 1\cdot5 + \frac{1}{16}) \div \frac{1}{8}}{(1\frac{8}{9} \times \frac{1}{2} \times 296 \times \frac{1}{10} \div \frac{1}{8}) \div 9472947}$$
5. Divide 97 lb. 3 oz. 4 dr. 1 scr. 17 grs. by 9 lb. 7 oz. 7 dr. 2 scr.
6. A wall is to be built 15 yards long, 7 feet high, and 13 in. thick, with a doorway 6 ft. high and 4 ft. wide ; how many bricks will it require, the solid content of each being 108 cubic inches ?
7. Multiply 9 ft. 6' 4" 7''' by 11 ft. 7' 9" 11'''.
8. Find the value of $\frac{4\frac{2}{7} + \frac{8}{9} - \sqrt{2}}{\frac{1}{3} \text{ of } \frac{2}{3} + \frac{1}{4} \text{ of } \frac{2}{3}}$.
9. Reduce 782436 pints to bushels, &c.
10. Find the least common multiple of 77, 42, 27, 21, 33, 14, 7, 11, 63, and 30.
11. Divide 36787942 by 28e4 in the *duodecimal* scale. Also change 3762814 from the *nonary* to the decimal scale.
12. How many divisors has the number 150528 ?
13. Find the value of $\cdot1234625$ of 2 weeks and 2 days.
14. Multiply 27 lb. 4 oz. 3dr., avoirdupois, by 728 $\frac{1}{2}$.
15. Add together \$98\cdot17, \$42\cdot29, £16 3s. 8 $\frac{1}{2}$ d., \$97\cdot19, \$127\cdot87 $\frac{1}{2}$, and from their sum subtract £67 17s. 7 $\frac{1}{2}$ d.
16. Reduce $\cdot8$, $\cdot76$, $\cdot9123$, and $\cdot003327$ to their equivalent vulgar fractions.

17. Take the number 704 and by removing the decimal point. (1) Make it 10000 times greater; (2) make it 1000000 times less; (3) make it billions; (4) make it hundredths of billions; (5) make it tenths of millionths; (6) make it hundredths.

$$[\{(2\frac{1}{2} \times 5 \text{ of } 1\frac{1}{2}) + 9\frac{1}{2} + \cdot\dot{0}9 + \frac{2}{3}\frac{3}{7}\} - 11\frac{6}{7}] \div (\frac{1}{8} + \text{of } \cdot 16)^*$$

$$[(\cdot 7632763 \times 11) \times \frac{1}{2} \text{ of } \frac{9}{10}\frac{1}{2}] \times (\frac{1}{2} \text{ of } \cdot 2 \text{ of } \cdot 3 \text{ of } \cdot 25 \text{ of } 96) \div \cdot 2$$

18. Reduce $\frac{1}{2}$ of $\cdot 6732467 \div \frac{1}{3}$
19. Divide £550 3s. 1½d. among 4 men, 6 women, and 8 children, giving to each man double of a woman's share; and to each woman triple of a child's.
20. Add together $16\frac{7}{11}$, $19\frac{2}{3}$, $23\frac{1}{4}$, and $120\frac{5}{6}$.
21. Write down all the divisors of 8100.
22. Find the G. C. M. of 2691, 11817 and 9828.
23. Find the exact length of the lunar month which contains 2551443 seconds, and of the solar year, which contains 31556928 seconds.
24. How many times will a carriage wheel turn in going from Toronto to Hamilton, a distance of 38 miles, the circumference of the wheel being 14 feet 11 inches?
25. What is the weight of the water contained in a rectangular cistern 11 feet wide, 13 feet long and 15 feet deep, and how many gallons of water does it contain?

NOTE.—A cubic foot of water weighs 62½ lbs. and a gallon weighs 10 lbs.

26. Reduce £73 17s. 11½d. to dollars and cents.
27. From $93\frac{1}{4}$ take $76\frac{1}{2}$ and divide the result $\frac{1}{2}\frac{7}{8}$.
28. Find the value of $\frac{5\frac{1}{2} \div \frac{2}{3}}{1\frac{1}{2} \text{ of } \frac{4}{9} \div 10\frac{1}{4}} \times \frac{3}{8} \text{ of } \frac{1\frac{1}{2} \text{ of } 4\frac{1}{2}}{13\frac{1}{8} \text{ of } 5\frac{1}{2}}$.
29. Transform 91342 *undenary* into the *quinary*, *duodenary* and *binary* scales and prove the results by reducing all four numbers to the decimal scale.
30. What are the prime factors of 7680?
31. Reduce 72 miles, 3 fur., 7 per., 2 yds., 1 ft., 7 in. to lines.
32. Find the price of 97 pairs of gloves at 47 cents per pair.
33. What is the worth of a pile of cord wood 73 feet long, 4 feet wide and 11 feet high, at \$3·62½ per cord?
34. Divide $93\cdot 72\dot{3}$ by $29\cdot 41\dot{7}3$.
35. How many bushels of oats are there in 73429 lbs.?
36. What is the worth of 719630 lbs. of wheat at \$1·80 per bushel?
37. Add together \$72·14 and \$93·76; multiply the sum by 0·47 and divide the product equally among 11 persons.
38. Find the G. C. M. of 21389 and 180781.

* These questions though apparently difficult are not so in reality—they are designed for exercise in canceling, and do not require much work.

39. Reduce $\frac{7}{11}$, $\frac{4}{3}$, $\frac{2}{3}$, $\frac{8}{33}$, $\frac{1}{4}$, $\frac{7}{10}$ and $\frac{1}{2}$ to equivalent fractions, having a common denominator.
40. Purchased 17 yards of cotton at 11 cents per yard, 19 yards of ribbon at $37\frac{1}{2}$ cents a yard, $14\frac{1}{2}$ yards of silk at \$2.17 a yard, a parasol \$4.75, a bonnet \$11.50, 67 yards of sheeting at 27 cents a yard, 15 yards of French merino at \$1.37 $\frac{1}{2}$ a yard, and trimmings \$7.93. Require the amount of my bill.

SECTION V.

RATIO AND PROPORTION.

1. Two numbers having the same unit may be compared with one another in two ways :

1st. By considering *how much greater or less* one is than the other ; and

2nd. By considering *how many times* one contains the other.

2. Ratio is the relation which one number bears to another with respect to magnitude, when the numbers are compared by considering, not *how much greater or less* one is than the other, but *how many times* or *parts of a time* one contains the other. Hence :

The ratio of two numbers is the *quotient* arising from the division of one by the other.

Thus the ratio of 18 to 6 is 3, since $18 \div 6 = 3$, the ratio of 7 to 21 is $\frac{1}{3}$, since $7 \div 21 = \frac{7}{21} = \frac{1}{3}$.

3. The ratio of one number to another, when measured with respect to their *difference*, is sometimes called *arithmetical ratio*, to distinguish it from the ratio considered as in (Art. 2), which is called *geometrical ratio*.

In the following pages, whenever the term ratio is used, *geometrical ratio* is meant ; we shall use the term *difference* in place of *arithmetical ratio*.

4. Since ratio simply expresses the quotient arising from the division of one number by another, and since (Art. 66, Sect. II.) we have three ways of indicating division, it follows that we have three ways of expressing the ratio of one number to another.

Thus the ratio of 9 to 4 is expressed either by $9 \div 4$, or by $\frac{9}{4}$, or by 9 : 4.

The ratio of 7 to 13 is indicated either by $7 \div 13$, or by $\frac{7}{13}$, or by 7 : 13.

5. Ratio can exist only between numbers of the same kind.

Thus it is obvious that no comparison with respect to magnitude can be made between 6 *hours* and 11 *pounds*, or between 19 *days* and 16 *miles*, &c. i.e., these numbers are not of the same kind, and therefore no ratio can exist between them.

6. Numbers are of the same kind when they are of the same denomination, or when they have the same unit, or when one can be multiplied so as to exceed the other.

7. The two given numbers which constitute the ratio are called the *terms of the ratio*; when spoken of together they are called a *couplet*.

8. The first term of a couplet is called the *antecedent*; the last term, the *consequent*.

When the ratio is expressed in the form of a fraction, the numerator is the antecedent and the denominator the consequent.

9. Ratio is either *direct* or *inverse*, *simple* or *compound*.

10. A Direct Ratio is that which arises from the division of the antecedent by the consequent.

11. An *Inverse* or *Inverted Ratio* is that which arises from the division of the consequent by the antecedent.

Thus the inverse ratio of 15 to 3 is $3:15$, or $\frac{3}{15}$, or $3 \div 15$, or $\frac{1}{5}$.

12. An Inverse Ratio is sometimes called a reciprocal ratio.

Thus the reciprocal ratio of 15 to 3 is $3:15$ or $\frac{3}{15} = \frac{1}{5}$ = inverse ratio of 15 to 3.

13. The reciprocal of a quantity is unity divided by that quantity.

Thus the reciprocal of 8 is $\frac{1}{8}$; of 11, $\frac{1}{11}$; of $\frac{7}{2}$, $\frac{2}{7}$; of $\frac{8}{13}$, $\frac{13}{8}$; of $\frac{1}{9}$, 9: of $\frac{6}{13}$, $\frac{13}{6}$, &c.

14. When the direct ratio of two numbers is expressed by points, the inverse or reciprocal ratio is expressed by inverting the order of the terms; when by a fraction, by inverting the fraction.

15. A Simple Ratio is one that has but *one* antecedent and *one* consequent.

Thus 9:3, 7:11, 18:2, &c., are simple ratios.

16. A Compound Ratio is a ratio produced by *compounding* or multiplying together the corresponding terms of two or more simple ratios.

Thus, the simple ratio of 9:3 is 3.

the simple ratio of 24:2 is 12.

The ratio compounded of these is 216:6 = 36.

17. It must be distinctly remembered that a compound ratio is of the same nature as any other ratio, and, like a simple ratio, consists of one antecedent and one consequent. The term compound ratio is used merely to indicate the *origin* of the ratio in particular cases.

18. Ratios are compounded by multiplying together all the antecedents for a new antecedent, and all the consequents for a new consequent.

Thus, the ratio compounded of 2:7, 2:3, 5:11, and 4:3 is $2 \times 2 \times 5 \times 4 : 7 \times 3 \times 11 \times 3$ or 80:693.

EXERCISE.

- | | |
|----------------------------------|------------------------------|
| 1. What is the ratio of 27 to 3? | <i>Ans.</i> 9. |
| 2. What is the ratio of 7 to 11? | <i>Ans.</i> $\frac{7}{11}$. |
| 3. What is the ratio of 9 to 27? | <i>Ans.</i> $\frac{1}{3}$. |
| 4. What is the ratio of 42 to 5? | <i>Ans.</i> $8\frac{2}{5}$. |
| 5. What is the ratio of 72 to 6? | <i>Ans.</i> 12. |

Required the ratio of the following numbers:—

- | | | | |
|----------------|-------------------------------|-----------------------------|-------------------------------|
| 6. 5 to 25. | <i>Ans.</i> $\frac{1}{5}$. | 13. \$17 to \$8.50. | <i>Ans.</i> 2. |
| 7. 49 to 7. | <i>Ans.</i> 7. | 14. \$93 to \$31. | <i>Ans.</i> 3. |
| 8. 83 to 7. | <i>Ans.</i> $11\frac{6}{7}$. | 15. 14 bus. to 2 pks. | <i>Ans.</i> 28. |
| 9. 187 to 11. | <i>Ans.</i> 17. | 16. 40 m. to 12 fur. | <i>Ans.</i> $26\frac{2}{3}$. |
| 10. 19 to 152. | | 17. 24 lb. to 12 oz. | |
| 11. 23 to 299. | | 18. 17 shillings to £51. | |
| 12. 147 to 21. | | 19. 16 acres to 30 sq. per. | |

Required the inverse ratio of the following numbers:—

- | | | | |
|----------------|-----------------------------|----------------------------------|-------------------------------|
| 20. 7 to 21. | <i>Ans.</i> $\frac{3}{7}$. | 27. 6 days to 4 weeks. | <i>Ans.</i> $\frac{4}{3}$. |
| 21. 12 to 2. | <i>Ans.</i> $\frac{1}{6}$. | 28. 11 min. to 30 sec. | <i>Ans.</i> $\frac{1}{2}$. |
| 22. 27 to 6. | <i>Ans.</i> $\frac{3}{2}$. | 29. 4 lbs. to 12 oz. | <i>Ans.</i> $\frac{1}{3}$. |
| 23. 9 to 36. | <i>Ans.</i> $\frac{1}{4}$. | 30. 3 qts. to 43 gals. | <i>Ans.</i> $57\frac{1}{3}$. |
| 24. 19 to 57. | | 31. 70 per. to 2 miles. | |
| 25. 81 to 9. | | 32. 7 Flem. ells to 9 Eng. ells. | |
| 26. 187 to 17. | | 33. 11 oz. to 68 scruples. | |

Required the reciprocal ratio of the following numbers:—

- | | | | |
|--------------------------------------|---|--|------------------------------|
| 34. 7 to 42. | <i>Ans.</i> $\frac{1}{6}$: $\frac{1}{7}$ = 6. | 39. $\frac{1}{24}$ to $\frac{1}{36}$. | <i>Ans.</i> $\frac{3}{4}$. |
| 35. $\frac{1}{2}$ to $\frac{1}{4}$. | <i>Ans.</i> 8 : 2 = 4. | 40. 72 to 18. | <i>Ans.</i> $\frac{1}{4}$. |
| 36. 42 to 28. | <i>Ans.</i> $\frac{1}{2}$: $\frac{1}{8}$ = $\frac{3}{4}$. | 41. 512 to 32. | <i>Ans.</i> $\frac{1}{16}$. |
| 37. 17 to 68. | | 42. $\frac{1}{4}$ to $\frac{7}{8}$. | |
| 38. 19 to 17. | | 43. $\frac{3}{4}$ to $\frac{1}{2}$. | |

Required the ratios compounded of the following ratios:—

- | | |
|---|-------------------------|
| 44. 2 to 3, 5 to 7 and 1 to 7. | <i>Ans.</i> 10 to 147. |
| 45. 8 to 6 and 17 to 3. | <i>Ans.</i> 136 to 18. |
| 46. 9 to 8, 7 to 6, 5 to 6, 4 to 3 and 2 to 1. | <i>Ans.</i> 2520 : 864. |
| 47. 1 to 7, 1 to 3, 3 to 1 and 5 to 1. | <i>Ans.</i> 15 : 21. |
| 48. 2 to 5, 3 to 7, 4 to 5, 21 to 2 and 1 to 9. | <i>Ans.</i> 504 : 3150. |

19. Since the antecedent of a couplet is a dividend, the consequent a divisor, and the ratio the quotient, it follows from the principles established in Arts. 79–84, Sect. II., that:—

1st. Multiplying the antecedent of a couplet or dividing the consequent by any number multiplies the ratio by that number.

Thus the ratio of 28 to 112 = $\frac{1}{4}$.

The ratio of 28×3 to $112 = \frac{3}{4} = \frac{1}{4} \times 3$ = three times the ratio of 28 to 112.

2nd. Dividing the antecedent of a couplet or multiplying the consequent by any number divides the ratio by that number.

Thus the ratio of 64 to 16 = 4.

The ratio of $64 \div 2$ to $16 = 32 : 16 = 2 = 4 \div 2$ = half the ratio of 64 to 16.

3rd. Multiplying or dividing both antecedent and consequent of a couplet by the same number does not alter the value of the ratio.

Thus the ratio of 18 to 6 is 3.

The ratio of $18 \times 7 : 6 \times 7 = 126 : 42 = 3 =$ ratio of $18 \div 2 : 6 \div 2 = 9 : 3$.

20. Since any number of ratios to be compounded together may be expressed as fractions and then compounded by the rule for multiplication of fractions (Art. 45, Sect. IV.) it follows that:—

When several ratios are to be compounded together we may, before multiplying the corresponding terms together, cancel any factor that is common to an antecedent and a consequent.

EXAMPLE 49.—Compound together 4 : 17, 34 : 55, 11 : 2, 13 : 7, and 21 : 65.

OPERATION.

$$\left. \begin{array}{l} 4 : 17 \\ 2 \\ 34 : 55 \\ 11 : 2 \\ 13 : 7 \\ 3 \\ 21 : 65 \\ 5 \end{array} \right\} = 4 \times 3 : 5 \times 5$$

or

$$12 : 25 \text{ Ans.}$$

EXPLANATION.—17 cancels 17 and reduces 34 to 2 and this 2 cancels 2, the third consequent; 11 reduces 55 to 5; 13 reduces 65 to 5 and 7 reduces 21 to 3. The only antecedents now left are 4 and 3 which multiplied together make 12, and the only remaining consequents are 5 and 5 which multiplied together make 25. The ratio 12 to 25 is therefore the ratio compounded of all the given ratios.

EXAMPLE 50.—Compound the following ratios:—

OPERATION.

$$\left. \begin{array}{l} 7 : 16 \\ 3 \\ 24 : 3 \\ 9 : 49 \\ 2 \\ 14 : 11 \\ 2 \\ 22 : 13 \end{array} \right\} = 9 \times 2 : 13$$

or

$$18 : 13 \text{ Ans.}$$

EXAMPLE 51.—Find the ratio compounded of the following ratios:—

OPERATION.

$$\left. \begin{array}{l} 1 : 7 \\ 16 : 22 \\ 22 : 14 \\ 7 \\ 22 : 31 \\ 11 \\ 25 : 64 \\ 4 \end{array} \right\} = 1 \quad 4 \text{ Ans.}$$

EXERCISE.

52. Find the ratio compounded of 9 : 16, 25 : 31, 341 : 18 and 48 : 100. *Ans.* 33 : 8.
 53. Find the ratio compounded of 18 : 25, 7 : 9, 11 : 12, and 91 : 49. *Ans.* 143 : 150.
 54. Find the ratio compounded of 1 : 2, 2 : 3, 3 : 4, 4 : 5, 5 : 6 and 7 : 11. *Ans.* 7 : 66.
 55. Find the ratio compounded of 2 : 5, 8 : 11, 14 : 17 and 187 : 112. *Ans.* 2 : 5.
 56. Find the ratio compounded of 3 : 5, 7 : 9, 11 : 13, 15 : 17 and 19 : 21. *Ans.* 209 : 663.

21. If the antecedent of a couplet be *equal* to the consequent, the ratio is equal to 1 and is called a *ratio of equality*.

If the antecedent be greater than the consequent the ratio is greater than 1 and is called a *ratio of greater inequality*.

If the antecedent be less than the consequent the ratio is less than 1, and is called a *ratio of less inequality*.

Thus the ratio of 7 : 7 = 1 is a ratio of equality.

The ratio of 7 : 2 = $3\frac{1}{2}$ is a ratio of greater inequality.

The ratio of 7 : 14 = $\frac{1}{2}$ is a ratio of less inequality.

EXERCISE.

In examples 1-43 of the foregoing Exercises point out which are ratios of *greater* and which ratios of *less inequality*.

22. Ratios are compared with one another by expressing them in the form of fractions—reducing these to their equivalent fractions having a common denominator and comparing the numerators.

Ratios may also be compared by actually dividing the antecedent by the consequent and thus ascertaining which gives the greatest quotient.

NOTE.—The latter method is usually the more convenient.

EXAMPLE 57.—Which is the greatest and which the least of the following ratios, viz : 3 : 4, 7 : 8, and 9 : 10.

By 1st Rule $\left. \begin{array}{l} 3 : 4 = \frac{3}{4} = \frac{30}{40} \\ 7 : 8 = \frac{7}{8} = \frac{35}{40} \\ 9 : 10 = \frac{9}{10} = \frac{36}{40} \end{array} \right\} \text{Hence } 9 : 10 \text{ is greatest and } 3 : 4 \text{ least.}$

By 2nd Rule $\left. \begin{array}{l} 3 : 4 = 3 \div 4 = .75 \\ 7 : 8 = 7 \div 8 = .875 \\ 9 : 10 = 9 \div 10 = .9 \end{array} \right\} \text{Hence } 9 : 10 \text{ is greatest and } 3 : 4 \text{ least.}$

EXAMPLE 58.—Compare together the following ratios, 7 : 8, 3 : 3 and 11 : 3 and 5 : 6.

$$\text{By 1st Rule } \left. \begin{array}{l} 7 : 8 = \frac{7}{8} = \frac{31}{36} \\ 2 : 3 = \frac{2}{3} = \frac{24}{36} \\ 11 : 13 = \frac{11}{13} = \frac{30}{36} \\ 5 : 6 = \frac{5}{6} = \frac{30}{36} \end{array} \right\} \text{Hence } 7 : 8 \text{ is the greatest and } 2 : 3 \text{ is the least.}$$

$$\text{By 2nd Method } \left. \begin{array}{l} 7 : 8 = 7 \div 8 = .875 \\ 2 : 3 = 2 \div 3 = .6 \\ 11 : 13 = 11 \div 13 = .846153 \\ 5 : 6 = 5 \div 6 = .83 \end{array} \right\} \text{Hence } 7 : 8 \text{ is the greatest and } 2 : 3 \text{ the least.}$$

EXERCISE.

59. Point out which is greatest and which least of the ratios
7 : 4, 6 : 3, 17 : 8, and 11 : 5.

Ans. 11 : 5 is greatest and 7 : 4 least.

60. Point out which is greatest and which least of the ratios
16 : 9, 10 : 3, 7 : 2, and 8 : 3.

Ans. 7 : 2 is greatest and 16 : 9 least.

61. Point out which is greatest and which least of the ratios
7 : 33, 11 : 49, 16 : 71, and 21 : 106.

Ans. 16 : 71 is the greatest and 21 : 106 least.

23. If the terms of two or more couplets, having the same ratio, be added together, the resulting couplet will have the same ratio.

Thus, the ratio of 6 : 2 = 3, the ratio of 21 : 7 = 3, and the ratio of 33 : 11 = 3, and the ratio 6+21+33 : 2+7+11, that is, of 60 to 20 is also 3.

That is, if 6 : 2 = 21 : 7 = 33 : 11, then 6+21+33 : 2+7+11 = 3 : 2.

24. If from the terms of any couplet the terms of another couplet having the *same ratio* be subtracted, then the resulting couplet will have the same ratio.

Thus, the ratio of 35 to 5 is 7, and the ratio of 14 to 2 is 7. So also the ratio of 35-14 : 5-2, that is, of 21 : 3 is 7, or, if 35 : 5 = 14 : 2, then 35-14 : 5-2 = 35 : 5.

25. A ratio of *greater inequality* is *diminished* by adding the same number to both terms.

Thus, the ratio of 48 : 8 = 6.

The ratio of 48+12 : 8+12 or 60 : 20 = 3 which is less than ratio 48 : 8.

26. A ratio of *less inequality* is *increased* by adding the same number to both terms.

Thus, the ratio of 8 : 48 = $\frac{1}{6}$.

The ratio of 8+12 : 48+12 or 20 : 60 = $\frac{1}{3}$ which is greater than ratio of 8 : 48.

PROPORTION.

27. Proportion is an equality of ratios.

Thus, the ratios 15 : 3 and 25 : 5 constitute a proportion, since $15 : 3 = 5 = 25 : 5$.

28. The terms of the two couplets are called proportionals.

29. Proportion may be expressed in two ways,

1st. By placing =, the sign of equality, between the ratios.

2nd. By placing four points, thus ::, between the two ratios.

Thus, we may express the proportion existing between 15, 3, 25, and 5 by $15 : 3 = 25 : 5$, or by $15 : 3 :: 25 : 5$.

We read either of them by saying the ratio of 15 to 3 equals the ratio of 25 to 5; or simply 15 is to 3 as 25 is to 5.

NOTE.—The sign :: is supposed to be derived from =, the sign of equality the four points being merely the *extremities* of the lines.

30. In every proportion there must be *four terms*, since there must be two couplets, and each couplet consists of two terms.

31. When three numbers constitute a proportion, one of them is repeated so as to form two terms.

Thus, if 18, 6, and 2 are proportionals.

$18 : 6 :: 6 : 2$.

In this case the 6, i. e., the term repeated, is called the *middle term* or a *mean proportional* between the other two numbers.

The 2 is called the *third term* or a *third proportional* to the other two numbers.

32. It is important to remember the distinction between *ratio* and *proportion*.

A ratio consists of two *terms*, an antecedent and a consequent.

A proportion consists of two couplets or four terms.

One ratio may be greater or less than another.

One proportion cannot be greater or less than another, since equality does not admit of degrees.

33. The outer terms of a proportion are called the *extremes*, and the two intermediate ones, the *means*.

Thus, in the proportion $3 : 17 :: 21 : 119$.

3 and 119 are the extremes.

17 and 21 are the means.

34. If four quantities be proportionals, the product of the extremes is equal to the product of the means.

This, if $6 : 11 :: 18 : 33$. Then $6 \times 33 = 11 \times 18$.

This may be established in the following manner:— $6 : 11 = \frac{6}{11}$, and $18 : 33 = \frac{18}{33}$, and since $6 : 11 :: 18 : 33$, $\frac{6}{11} = \frac{18}{33}$ (Art. 27.) Now, since multiplying equals by the same number does not destroy their equality, if we multiply these fractions by 11 we get $6 = \frac{18 \times 11}{33}$; and multiplying each of these by 33, we have $6 \times 33 = 18 \times 11$; but 6 and 33 are the extremes and 18 and 11 are the means, therefore in any geometrical proportion the product of the extremes equals the product of the means.

The same fact may be established more generally as follows:—

Let a, b, c and d be any four proportionals whatever.

Then $a : b :: c : d$

But $a : b = \frac{a}{b}$ and $c : d = \frac{c}{d}$

Therefore $\frac{a}{b} = \frac{c}{d}$ — Multiplying each of these equals by $c \times d$, we have

$a \times d = b \times c$. But a and d are the extremes and b and c are the means. Therefore, &c.

35. This principle then may be considered the *test* of a geometrical proportion. If the product of the extremes equals the product of the means, the four quantities are proportional; if the products are not equal, the numbers are not proportional.

36. It follows from Art. 34 that:—

1st. If the product of the means be divided by one extreme, the quotient will be the other extreme.

2nd. If the product of the extremes be divided by one mean, the quotient will be the other mean.

and hence,

3rd. If any three terms of a proportion be given, the fourth may be found thus:

$$1st\ term = \frac{2nd\ term \times 3rd\ term}{4th\ term.}$$

$$2nd\ term = \frac{1st\ term \times 4th\ term}{3rd\ term.}$$

$$3rd\ term = \frac{1st\ term \times 4th\ term}{2nd\ term.}$$

$$4th\ term = \frac{2nd\ term \times 3rd\ term}{1st\ term.}$$

EXAMPLE 1.—What is the fourth proportional to 7, 11 and 35 ?

$$4th\ term = \frac{2nd\ term \times 3rd\ term}{1st\ term.} = \frac{11 \times 35}{7} = 55\ Ans.$$

EXAMPLE 2.—The first, second and fourth terms of a proportion are 9, 16 and 128. Required the third term.

$$3rd\ term = \frac{1st \times 4th}{2nd} = \frac{9 \times 128}{16} = 72\ Ans.$$

EXERCISE.

3. The second, third and fourth terms of a proportion are 17, 11 and $93\frac{1}{2}$. What is the first term? *Ans.* 2.
 4. The first, third, and fourth terms of a proportion are 21, 63 and 39. Required the second term. *Ans.* 13.
 5. The first three terms of a proportion are 2, 3 and 7. What is the fourth term? *Ans.* $10\frac{1}{2}$.
 6. The last three terms of a proportion are 91, 88 and 104. Required the first term. *Ans.* 77.
 Find the fourth proportional to
 7. 4 yds. 18 yds. and \$96. *Ans.* \$432.
 8. 5 lb. 2 lb. and \$3.75. *Ans.* \$1.50.
 9. 1 cwt. 215 cwt. and \$7.50. *Ans.* \$1612.50.
 10. 6 miles, 1 mile and 27 shillings. *Ans.* 4s. 6d.
 11. 10 lb. 150 lb. and £6 3s. 9d. *Ans.* £92 16s. 3d..
 12. 4 days, 27 days and \$100. *Ans.* \$675.

37. It will be useful to remember the following properties of a Geometrical proportion. As the proofs are given in every common work on Algebra, it has not been thought advisable to insert them here. a, b, c and d stand for any four proportionals whatever.

| | |
|---|--|
| If $a : b :: c : d$ | Or if $15 : 6 :: 10 : 4$ |
| Alternately $a : c :: b : d$ | $15 : 10 :: 6 : 4$ |
| Inversely $b : a :: d : c$ | $6 : 15 :: 4 : 10$ |
| By Composition $a + b : b :: c + d : d$ | $15 + 6 : 6 :: 10 + 4 : 4$, or $21 : 6 :: 14 : 4$ |
| By Division $a - b : b :: c - d : d$ | $15 - 6 : 6 :: 10 - 4 : 4$, or $9 : 6 :: 6 : 4$ |
| By Conversion $a : a + b :: c : c + d$ | $15 : 15 + 6 :: 10 : 10 + 4$, or $15 : 21 :: 10 : 14$ |
| Or $a : a - b :: c : c - d$ | $15 : 15 - 6 :: 10 : 10 - 4$, or $15 : 9 :: 10 : 6$ |

38. Proportion in Arithmetic is usually divided into simple, compound and conjoined.

SIMPLE PROPORTION.

39. Simple Proportion is frequently called the Rule of Three, because when *three* terms are given, by means of them a fourth may be found. It is also sometimes called the Golden Rule from its extensive utility.

40. EXAMPLE 13.—If 16 barrels of flour cost \$112, what will 129 barrels cost?

In this and every other question in Simple Proportion there are two ratios, one of which is perfect (*i.e.* has both terms given) and the other imperfect, and from the nature of proportion we know that these two ratios must be both of the same kind, that is, they must be both ratios of *greater inequality* or both ratios of *less inequality*.

Now in the above example, the ratio of \$112 to the answer is a ratio of *less inequality* since it is evident that, if 16 barrels cost \$112, 129 barrels will cost more. Therefore the other ratio is also a ratio of *less inequality* and must be written 16 : 129.

And since the ratios are equal,
barrels. dollars.
16:129::112:Ans.

$$\text{Also (Art. 36) } Ans. = \frac{112 \times 129}{16} = \$903.$$

PROOF.—Set 903 in the fourth place, thus:
16:129::112:903
and see if the product of extremes = product of means (Art. 35.)
 $16 \times 903 = 14448 = 129 \times 112.$

From the preceding illustrations and principles we deduce for Simple Proportion the following general

RULE.

Set the given term of the imperfect ratio in the third place, and the letter x , to represent the answer, in the fourth.

Then, if, by the nature of the question, the ratio of the third term to the answer is a ratio of greater inequality, make the remaining ratio a ratio of greater inequality also; but if the ratio of the third term to the answer be a ratio of less inequality, make the other ratio a ratio of less inequality also.

Lastly, (Art. 36,) multiply the second and third terms together, divide the product by the first term, and the quotient will be the answer in the same denomination as the third term.

PROOF.—Multiply the first term and the answer together, and, if the product is equal to the product of the second and third terms, the work is correct. (Art. 35.)

EXAMPLE 14.—If a man can walk 155 miles in 12 days, how many miles can he walk in 60 days?

Here the imperfect ratio is 155 miles to x , and, in order to ascertain whether it is a ratio of greater or less inequality, we have merely to ask the following simple question—If a man can walk 155 miles in 12 days, can he walk more or less in 60 days? Evidently more. Therefore the ratio of 155 : x is a ratio of less inequality; or, in other words, the antecedent must be the least of the two numbers, and the statement is

$$\text{days. miles.} \\ 12:60::155:x.$$

Whence the answer = $\frac{60 \times 155}{12} = 775$ miles.

41. Since the second and third terms multiplied together, constitute a dividend, and the first term is a divisor, it is manifest, from the principles of division (Arts. 79-84, Sect. II.), that we may cancel any factor that is common to the first term and either of the other terms.

Thus in the last example we have 12:60::155: x and, dividing the first and second by 12, we get 1:5::155: x and $155 \times 5 = 775$ Ans.

EXAMPLE 15.—If 96 bushels of wheat cost \$128, what will 15 bushels cost?

As the answer to the question must be in dollars, the imperfect ratio is \$128 : x , and from the nature of the question, we know that 15 bushels will

cost less than 96 bushels, we therefore place 15, the smaller of the remaining terms, in the *second place*, and the other term, 96, in the *first place*. Hence, the statement is 96 : 15 bushels :: \$128 : x

OPERATION.

$$\begin{array}{r} \text{bush.} \quad \$ \\ 96 : 15 :: 128 : x \end{array}$$

Here 32 reduces 96 to 3 and 128 to 4, and 3 cancels 3 and reduces 15 to 5.

$$\$ 5 \times 4 = \$20 \text{ Ans.}$$

The teacher would do well to insist upon his pupils performing all questions in Proportion by analysis.

Thus, to solve the last question, we begin as follows: If 96 bushels cost \$128, 1 bushel will cost $\frac{1}{96}$ of \$128, or \$1 $\frac{33}{8}$. Then if 1 bushel cost \$1 $\frac{33}{8}$, 15 bushels will cost 15 times as much, which is \$20.

EXAMPLE 16.—If 27 men can mow 60 acres of grass in a day, how many acres can 93 men mow?

OPERATION.

men. acres.

$$27 : 60 :: 93 : x$$

$$\begin{array}{r} 9 \quad 31 \quad 20 \\ 3 \end{array}$$

3

$$\frac{31 \times 20}{3} = 206\frac{2}{3} \text{ acres Ans.}$$

Here the imperfect ratio is 60 : x acres, and since 93 men will evidently mow more than 27 men, we make 93 the *second term* and 27 the *first*. Hence the statement is 27 : 93 : 60 : x . Then 3 reduces 27 to 9 and 93 to 31, and 3 again reduces 9 to 3 and 60 to 20, and the answer is equal to 31 multiplied by 20 and divided by 3.

This question may be thus performed by analysis :

If 27 men mow 60 acres a day, 1 man will mow $\frac{1}{27}$ of 60 acres, or 2 $\frac{2}{3}$ acres ; 93 men will therefore mow 93 times 2 $\frac{2}{3}$ acres = 206 $\frac{2}{3}$ Ans.

EXERCISE.

17. If 11 baskets of peaches cost \$13.42, what will 87 baskets cost? *Ans.* \$106.14.
18. If 28 cords of wood cost \$266, what will 25 cords cost? *Ans.* \$237.50.
19. If a man receives \$29.20 for 16 days' work, for how many days should he work for \$83.60? *Ans.* 45 $\frac{4}{9}$ days.
20. If 16 bags of potatoes are sold for \$12.80, what will 156 bags bring? *Ans.* \$124.80.
21. If a stick 7 feet long cast a shadow of 5 feet, what will be the height of a tree which casts a shadow of 112 feet long? *Ans.* 156 $\frac{2}{7}$ feet.
22. If a stack of hay will feed 27 cows for 99 days, how long will it feed 55 cows? *Ans.* 48 $\frac{2}{3}$ days.
23. If 9 bushels of peas sow 5 acres, how many bushels will be required to sow 48 acres? *Ans.* 86 $\frac{2}{3}$ bushels.
24. If 3 men put up 73 perches of fencing in 2 days, how long will they take to put up 803 perches? *Ans.* 22 days.
25. If 176 pails of maple sap make 100 lbs. of sugar, how much sugar will 1128 pails make? *Ans.* 640 $\frac{1}{4}$ lbs.
26. If it cost \$20.88 to weave 108 yards of cloth, what will it cost to weave 465 yards? *Ans.* \$89.90.

27. If \$16 pay for the carriage of 72 barrels of flour, for the carriage of how many barrels will \$1278 pay? *Ans.* 5751 barrels.
28. If 11 men plough 165 acres in a week, how many acres would 3 men plough in the same time? *Ans.* 45 acres.
29. If 4 barrels of flour make 250 four-pound loaves of bread, how many such loaves will 67 barrels make?
Ans. 4187½ loaves.
30. If 190 bushels of apples make 16 barrels of cider, how many barrels of cider will 38 bushels of apples make?
Ans. 3½ barrels.
31. If 90 men can build a wall in 12 days, how many men could build it in 15 days? *Ans.* 72 men.
32. If 17 days' work pay for 2 barrels of flour, for how many barrels will 279 days' work pay? *Ans.* 32½ barrels.
33. If a train travel 27 miles per hour, how far will it travel in 24 hours? *Ans.* 648 miles.
34. If 7 cows make 30 lbs. of butter a week, how much may be expected from 23 cows? *Ans.* 98¼ lbs.

42. *If any of the terms contain fractions or mixed numbers, apply the rules in Section IV.*

EXAMPLE 35.—If $\frac{2}{3}$ of a basket of peaches cost $\frac{3}{4}$ of a dollar, how much will $\frac{3}{4}$ of a basket of peaches cost?

OPERATION.

$\frac{2}{3} : \frac{3}{4} :: \frac{3}{4} : x$. Therefore answer = $\frac{3}{4} \times \frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{2} = 19\frac{3}{8}$ cents.

EXAMPLE 36.—If $\frac{3}{4}$ of a bushel cost $\frac{1}{4}$ of a pound, what will $\frac{1}{2}$ of a bushel cost?

OPERATION.

$\frac{3}{4} : \frac{1}{2} :: \frac{1}{4} : x$. Therefore answer = $\frac{1}{4} \times \frac{1}{2} \div \frac{3}{4} = \frac{1}{4} \times \frac{1}{2} \times \frac{4}{3} = \frac{1}{6} = 11s. 10\frac{2}{3}d.$

NOTE.—If the first term be a fraction, invert it and connect it to the others by the sign of multiplication.

EXERCISE.

37. If $\frac{3}{4}$ of a ship cost \$9750, what will $\frac{5}{8}$ cost? *Ans.* \$42000.
38. How much will $\frac{1}{2}$ of a yard come to if $\frac{2}{3}$ of a yard cost $\frac{5}{8}$ of a shilling? *Ans.* 2½d.
39. If \$7.49 pay for $\frac{7}{8}$ of a ton of coals, what will $8\frac{1}{2}$ tons cost? *Ans.* \$80.25.
40. If $5\frac{1}{2}$ yards of broadcloth cost \$28.42, what will $\frac{3}{4}$ of a yard come to? *Ans.* \$2.80.
41. If $\frac{1}{2}$ of a dollar pay for $\frac{1}{4}$ of a bag of apples, for what part of a bag will $\frac{1}{5}$ of a dollar pay? *Ans.* $\frac{2}{5}$ of a bag.
42. If \$100 stock is worth \$98½, what will \$472½ stock be worth? *Ans.* \$467.12½.

43. If $17\frac{2}{3}$ tons of hay last a certain number of horses $107\frac{1}{7}$ days, how many days will $11\frac{1}{7}$ tons last the same number of horses? *Ans.* $70\frac{2}{3}$ days.
44. If $22\frac{1}{3}$ cords of wood last as long as $15\frac{7}{3}$ tons of coal, how many cords of wood will last as long as $11\frac{2}{3}$ tons of coal? *Ans.* $16\frac{1}{3}$ cords of wood.
45. If $\frac{1}{2}$ of $\frac{2}{3}$ of $3\frac{1}{3}$ yards of broadcloth cost $\frac{2}{3}$ of $\frac{1}{7}$ of \$4 $\frac{2}{3}$, what will $\frac{2}{3}$ of $\frac{1}{2}$ of $\frac{2}{3}$ of a yard cost? *Ans.* $\$1\frac{1}{2}$ or \$0.669.
43. When the first and second terms are not of the same denomination or contain different denominations—

RULE.

Reduce both to the lowest denomination contained in either, and then apply the rule in Art. 40.

EXAMPLE 46.—If 11 bushels 2 pks. 1 gal. cost \$74, what will 76 bushels 1 pk. 1 gal. 1 qt. 1 pt. cost?

OPERATION.

The lowest denomination contained in either is pints.
 11 bush. 2 pks. 1 gal. : 76 bush. 1 pk. 1 gal. 1 qt. 1 pt. :: \$74 : x ; this reduced becomes 744 : 4891 :: \$74 : x.

$$\text{Ans. } \frac{\$74 \times 4891}{744} = \$486.47 \text{ +}$$

In this example 11 bush. 2 pk. 1 gal. = 744 pints and 76 bush. 1 pk. 1 gal. 1 qt. 1 pt. = 4891 pints.

EXERCISE.

47. What will 37 sq. yds. 4 ft. 120 in. of painting cost, if 9 sq. yds. 2 ft. cost \$3.50? *Ans.* \$14.245.
48. How much will 12 lb. 10 oz. of silver come to at \$1.25 per oz.? *Ans.* \$192.50.
49. If 10 yards of ribbon cost \$3.40, what will 3 yds. 2 qrs. cost? *Ans.* \$1.19.
50. If 15 oz. 12 dwt. 16 grs. cost \$3.80, what will 13 oz. 14 grs. cost? *Ans.* \$3.167.
51. What will 3 lb. 1 oz. 11 dwt. cost, if 12 lb. 6 oz. 4 dwt. cost \$600? *Ans.* \$150.
52. If a man can pump 54 barrels of water in 2 hrs. 46 min. 30 sec., in what time will he pump 24 barrels? *Ans.* 1 h. 14 min.
53. What will 73 yds. 3 qrs. 2 na. 1 in. of velvet cost, if 3 Flem. ells 2 qrs. 1 na cost £4 17s. 8½d.? *Ans.* £128 6s. 10½d.
54. If 4½ oz. avoirdupois cost 8½ shillings, what will 8¼ lbs. cost? *Ans.* £13 9s. 0½d.
55. In the copy of a work containing 327 pages, a remarkable passage commences at the end of the 156th page. On what page might it be expected to begin in a copy containing 400 pages? *Ans.* On the 191st page.

56. If the rent of 46 acres, 3 roods, and 14 perches be £100, what will be the rent of 35 acres, 2 roods, and 10 perches?
Ans. £75 18s. 6 $\frac{2}{3}$ $\frac{16}{3}$ $\frac{9}{7}$ d.
57. When A had travelled 68 days at the rate of 12 miles a day, B, who had travelled 48 days, overtook him. How many miles a day did B travel, allowing both to have started from the same place?
Ans. 17.
58. If 21 $\frac{1}{2}$ shillings pay for 16 $\frac{1}{2}$ lbs. of prunes, how many pounds can be bought for 32 $\frac{1}{2}$ shillings?
Ans. 24 $\frac{6}{10}$ $\frac{4}{8}$ lbs.
59. A ton of coal yields about 9000 cubic feet of gas; a street lamp consumes about 5, and an argand burner (one in which the air passes through the centre of the flame) 4 cubic feet in an hour. How many tons of coal would be required to keep 17493 street lamps, and 192724 argand burners in shops, &c., lighted for 1000 hours?
Ans. 95373 $\frac{1}{2}$.
60. The gas consumed in London requires about 50000 tons of coal per annum. For how long a time would the gas this quantity may be supposed to produce (at the rate of 9000 cubic feet per ton), keep one argand light, (consuming 4 cubic feet per hour) constantly burning?
Ans. 12842 years and 170 days.
61. Suppose 11270 lbs. of beef for a ship's use were to be cut up in pieces of 4 lb., 3 lb., 2 lb., 1 lb., and $\frac{1}{2}$ lb.—there being an equal number of each. How many pieces would there be of each?
Ans. 1073; and 3 $\frac{1}{2}$ lb. left.
62. The sloth does not advance more than 100 yards in a day. How long would it take to crawl from Toronto to Kingston, allowing the distance to be 180 miles?
Ans. 3168 days, or about 8 $\frac{2}{3}$ years.
63. Suppose that a greyhound makes 27 springs while a hare makes 25, and that their springs are of equal length. How many springs must the hound make to overtake the hare, if the latter has a start of 50 springs?
Ans. 675.

COMPOUND PROPORTION.

44. Compound Proportion is an equality between a *compound* ratio and a *simple* ratio.

Thus 7 : 11 compounded with 22 : 21 :: 34 : 51, is a compound ratio.
 Or 7 × 22 : 11 × 21 :: 34 : 51, and applying Art. 40 we have 7 × 22 × 51 = 11 × 21 × 34.

45. Compound Proportion is also called the Double Rule of Three. It enables us to obtain the answer by a single statement, although two or more proportions are contained in the question.

46. In Compound Proportion there are *three* or *more* ratios, one of which is imperfect and all the others perfect.

47. Let it be required to solve the following question: If 18 men dig a trench 30 yards long, in 24 days, by working 8 hours a day, how many men will dig a trench 60 yards long, in 64 days, working 6 hours a day?

Let us suppose the time to be the same in both cases, and this question becomes the same as the following:

If 18 men dig 30 yards of trench, how many men will dig 60 yards?

Here it is evident the answer will be the same fraction of 18 that 60 yards is of 30 yards; or, in other words, the required number of men = $\frac{2}{3}$ of 18 men.

Next let us take into account the number of days; but suppose they work the same number of hours per day in both cases.

The question then becomes: If $\frac{2}{3}$ of 18 men require 24 days to dig a trench, how many men will dig it in 64 days?

In this case it is plain that the answer will be the same fraction of $\frac{2}{3}$ of 18 men that 24 days is of 64 days; that is, the required number of men = $\frac{2}{3}$ of $\frac{2}{3}$ of 18 men.

Lastly, let us take into consideration the time worked each day.

The question then becomes: If $\frac{2}{3}$ of $\frac{2}{3}$ of 18 men dig a trench in a certain number of days, working 8 hours per day, how many men will dig it working 6 hours per day?

In this case the answer is obviously = $\frac{2}{3}$ of $\frac{2}{3}$ of $\frac{2}{3}$ of 18 men, or dividing these equal by 18. $\frac{\text{Answer}}{18} = \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$.

Or taking the reciprocals $\frac{18}{\text{Answer}} = \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2}$.

That is the ratio compounded of 6 : 8, 64 : 24, and 30 : 60 = ratio of 18 : Answer, or, $\left. \begin{array}{l} 30 : 60 \\ 64 : 24 \\ 6 : 8 \end{array} \right\} :: 18 : \text{Answer}$.

The answer is equal to the continued product of the third term, and all the second terms, divided by the continued product of all the first terms.

From the preceding principles and illustrations, we deduce the following general

RULE FOR COMPOUND PROPORTION.

Place that number which is of the same kind as the answer in the third term, and the letter *n* to represent the answer in the fourth term.

Then take the other numbers in pairs, or two of a kind, and arrange them as in simple proportion.

Finally, multiply together all the second terms and the third term, divide the result by the product of the first term, and the quotient will be the fourth term or answer required.

NOTE.—Since the third term and second terms multiplied together constitute a dividend, and the first terms multiplied together, a divisor, we may (Arts. 79–84, Sect. 11) cancel any factors that are common to any of the first terms and to the third term or any of the second terms.

EXAMPLE 1.—If 5 compositors, in 16 days, 11 hours long, can compose 25 sheets of 24 pages in each sheet, 44 lines in each page, and 40 letters in a line; in how many days, each 10 hours long, may 9 compositors compose a volume, to be printed in the same letter, consisting of 36 sheets, 16 pages to a sheet, 50 lines to a page, and 45 letters to a line?

| STATEMENT. | days. | SAME CANCELLED. |
|--------------------------|-----------|--|
| 9 comp. : 5 comp. | } | $9 : 5$ |
| 10 hours : 11 hours. | | $10 : 11$ |
| 25 sheets : 36 sheets. | | $25 : 36$ |
| 24 pages : 16 pages. | | $24 : 16$ |
| 44 lines : 50 lines. | | $44 : 50$ |
| 40 letters : 45 letters. | | $40 : 45$ |
| | :: 16 : x | $\left. \begin{array}{l} 4 \\ 4 \\ 11 \\ 3 \\ 10 \\ 3 \end{array} \right\} :: 16 : x.$ |
| | | <i>Ans.</i> $3 \times 4 = 12$ days. |

EXPLANATION.—The imperfect ratio is that of 16 days to an unknown number of days. We place this ratio to the right hand-side, as in Simple Proportion. Now we compare each pair of terms with this ratio, in order to decide whether they constitute a ratio of greater or less inequality. Thus, if 5 compositors require 16 days, will 9 compositors require more or less? Evidently less; therefore it is a ratio of greater inequality, and we must write it 9:5. Next, if 11 hours to the day require 16 days, will 10 hours to the day require more or less?—more; therefore we must write 10:11. Next, if 25 sheets require 16 days, will 36 days require more or less?—more; therefore we write 25:36. Next, if 44 lines to a page require 16 days, will 50 lines to a page require more or less?—more; therefore we write 44:50. Lastly, if 40 letters to a line require 16 days, will 45 letters to a line require more or less?—more; therefore we write 40:45.

The statement is now complete, and we cancel as follows; 5 cancels 5, the first consequent, and reduces 25, the third antecedent, to 5, and 5 cancels this 5, and reduces 50, the fifth consequent, to 10, and 10 cancels this 10 and 10, the second antecedent. Again, 9 cancels the first antecedent and reduces 36, the third consequent, to 4, and 4 cancels this 4 and reduces 44, the fifth antecedent, to 11, and 11 cancels this 11 and 11, the second consequent. Again, 8 reduces 24 to 3 and 16 to 2. 3 cancels this 3 and reduces 45 to 15. 2 cancels the 2 resulting from the 16 and reduces 40 to 20, and 5 reduces this 20 to 4 and the 15 resulting from 45 to 3. Lastly, 4 cancels this 4 and reduces 16, the third term, to 4. There remain but 3 and 4 which multiplied together make 12. *Ans.*

EXAMPLE 2.—If 24 men can saw 90 cords of wood in 6 days, when the days are 9 hours long, how many cords can 8 men saw in 36 days, when they are 12 hours long?

| STATEMENT. | cords. | SAME CANCELLED. |
|---------------------|------------|---|
| 24 men : 8 men. | } | $24 : 8$ |
| 6 days : 36 days. | | $6 : 36$ |
| 9 hours : 12 hours. | | $9 : 12$ |
| | :: 90 : x. | $\left. \begin{array}{l} 10 \\ 2 \\ 12 \end{array} \right\} :: 90 : x.$ |
| | | <i>Ans.</i> $10 \times 2 \times 12 = 240$ cords. |

Here the imperfect ratio is 90: *Ans.* If 24 men saw 90 cords, will 8 men saw more or less?—less; therefore it is a ratio of greater inequality, and we write 24:8. Next, if 6 days saw 90 cords of wood, will 36 days saw more or less?—more; therefore it is a ratio of less inequality, and we write 6:36. Lastly, if 9 hours per day saw 90 cords, will 12 hours per day saw more or less?—more; therefore it is a ratio of less inequality, and we write 9:12.

EXAMPLE 3.—If 248 men, in $5\frac{1}{2}$ days, of 11 hours each, dig a trench of 7 degrees of hardness, 232 $\frac{1}{2}$ yards long, 3 $\frac{1}{2}$ wide, and 2 $\frac{1}{2}$ deep; in how many days, of 9 hours long, will 24 men dig a trench of 4 degrees of hardness, 337 $\frac{1}{2}$ yards long, 5 $\frac{1}{2}$ wide, and 3 $\frac{1}{2}$ deep?

STATEMENT.

$$\left. \begin{array}{l} 24 : 248 \text{ men,} \\ 9 : 11 \text{ hours.} \\ 7 : 4 \text{ degrees.} \\ 232\frac{1}{2} : 337\frac{1}{2} \text{ yds. long.} \\ 3\frac{1}{2} : 5\frac{1}{2} \text{ yds. wide.} \\ 2\frac{1}{2} : 3\frac{1}{2} \text{ yds. deep.} \end{array} \right\} :: 5\frac{1}{2} \text{ days: Ans. or, } \left\{ \begin{array}{l} 24 : 248 \\ 9 : 11 \\ 7 : 4 \\ 465 : 675 \\ 1\frac{1}{3} : 2\frac{2}{3} \\ 3 : 7\frac{1}{2} \end{array} \right\} :: \frac{1}{2} : x.$$

The answer will be $(24^2 \times 11 \times 4 \times \frac{675}{2} \times 2^2 \times \frac{1}{2} \times \frac{11}{2}) \div (24 \times 9 \times 7 \times 465 \times \frac{1}{3} \times \frac{1}{2} \times 11 \times 3)$
 $= 24^2 \times 11 \times 4 \times \frac{675}{2} \times 2^2 \times \frac{1}{2} \times \frac{11}{2} \times \frac{1}{2} \times \frac{1}{9} \times \frac{1}{7} \times 465 \times \frac{1}{3} \times \frac{1}{2}$

CANCELLED.

$$\frac{248}{1} \times \frac{11}{1} \times \frac{4}{1} \times \frac{675}{2} \times \frac{28}{5} \times \frac{7}{2} \times \frac{11}{2} \times \frac{1}{24} \times \frac{1}{9} \times \frac{1}{7} \times \frac{2}{465} \times \frac{8}{11} \times \frac{3}{7} \\ = 4 \times 3 \times 11 = 132 \text{ days.}$$

EXERCISE.

- If 120 bushels of corn last 14 horses 56 days, how many days will 90 bushels last 6 horses? *Ans.* 98 days.
- If a wall of 28 feet high were built in 15 days by 63 men, how many men would build a wall 32 feet high in 8 days? *Ans.* 135 men
- If 1 lb. of thread make 3 yards of linen of $1\frac{1}{4}$ yards wide, how many pounds of thread would be required to make a piece of linen of 45 yards long and 1 yard wide? *Ans.* 12 lb.
- If 3 lb. of worsted make 10 yards of stuff of $1\frac{1}{4}$ yards broad, how many pounds would make a piece 100 yards long and $1\frac{1}{4}$ broad? *Ans.* 25 lb.
- If 12 horses in 5 days draw 44 tons of stones, how many horses would draw 132 tons the same distance in 18 days? *Ans.* 10 horses.
- If 27s. are the wages of 4 men for 7 days, what will be the wages of 14 men for 10 days? *Ans.* £6 15s.
- 3 masters, who have each 8 apprentices, earn \$144 in 5 weeks—each consisting of 6 working days. How much would 5 masters, each having 10 apprentices, earn in 8 weeks, working $5\frac{1}{2}$ days per week—the wages being in both cases the same? *Ans.* \$440.

11. If 6 shoemakers, in 4 weeks, make 36 pair of men's and 24 pair of women's shoes, how many pair of each kind would 18 shoemakers make in 5 weeks?
Ans. 135 pair of men's and 90 pair of women's shoes.
12. A wall is to be built of the height of 27 feet; and 9 feet high of it are built by 12 men in 6 days. How many men must be employed to finish the remainder in 4 days? *Ans.* 36.
13. If a footman travels 130 miles in 3 days, when the days are 14 hours long, in how many days of 7 hours each will he travel 390 miles? *Ans.* 18.
14. If the price of 10 oz. of bread, when the flour is 1s 10½d. per stone, is 1d., what must be paid for 3 lb. 12 oz. when the flour is 2s. 6d. per stone? *Ans.* 8d.
15. If 5 compositors in 16 days of 14 hours long, can compose 20 sheets of 24 pages in each sheet, 50 lines in a page, and 40 letters in a line; in how many days of 7 hours long may 10 compositors compose a volume to be printed in the same letter, containing 40 sheets, 16 pages in a sheet, 60 lines in a page, and 50 letters in a line? *Ans.* 32 days.
16. If 336 men, in 5 days of ten hours each, dig a trench of 5 degrees of hardness, 70 yards long, 3 wide, and 2 deep, what length of trench of 6 degrees of hardness, 5 yards wide, and 3 deep, may be dug by 240 men in 9 days of 12 hours each? *Ans.* 36 yards.
17. If a pasture of 16 acres will feed 6 horses for 4 months, how many acres will feed 12 horses for 9 months? *Ans.* 72 acres.
18. If 25 persons consume 300 bushels of corn in 1 year, how much will 139 persons consume in 7 years at the same rate? *Ans.* 1167 bushels.
19. If 32 men build a wall 36 feet long, 8 feet high, and 4 feet wide, in 4 days; in what time will 48 men build a wall 864 feet long, 5 feet high, and 3 feet wide? *Ans.* 30 days.
20. If a regiment of 679 soldiers consume 702 bushels of wheat in 336 days, how many bushels will an army of 22407 soldiers consume in 112 days? *Ans.* 7722 bushels.
21. If 12 tailors in 27 days can finish 13 suits of clothes, how many tailors in 19 days of the same length, can finish the clothes of a regiment of soldiers consisting of 494 men? *Ans.* 648 tailors.
22. If 17 head of cattle consume 5 acres 2 roods 10 perches of pasture in 30 days, how many acres would be consumed by 40 head in 51 days? *Ans.* 22 acres 1 rood.

23. If 180 bricks, 8 inches long, and 2 inches wide, are required for a walk 20 feet long, and 6 feet wide, how many bricks will be required for a walk 100 feet long and 4 feet wide?
Ans. 600 bricks.

CONJOINED PROPORTION.

48. Conjoined Proportion is a kind of Compound Proportion, in which the ratio of one of the terms to its corresponding term is made to depend on equivalencies among the intermediate terms of the proportion.

49. Conjoined Proportion is sometimes called the Chain Rule, from the peculiar manner in which the different pairs of terms are linked, as it were, together. It relates principally to exchanges between different countries, in respect to specie, weights, and measures, but is applicable to common business transactions.

50. EXAMPLE 1.—Suppose 7 yards of velvet in Toronto cost as much as 9 in Montreal, and 16 in Montreal as much as 24 in Paris, how many yards in Toronto will cost as much as 54 in Paris?

EXPLANATION.—This question may be stated as a problem in Compound Proportion as follows:

The imperfect ratio is 7 yards Toronto to an unknown number of yards, Toronto. Then, if 9 yards, Montreal, pay for 7 yards Toronto, will 16 yards pay for more or less?—more; therefore we write 9:16. Next if 24 yards Paris pay for a certain number $\left(\frac{16 \times 7}{9}\right)$ yards Toronto, will 54 yards Paris pay for more or less?—more; therefore we write the ratio 24:54. Now (Art. 47) the answer = $\frac{16 \times 54 \times 7}{9 \times 24}$; and it is evident that we may consider all the factors of the numerator as antecedents, and all the factors of the denominator as consequents, and then make the statement thus:

STATEMENT.

| | | | | |
|----|--------------|---|----|----------------|
| 7 | yds. Toronto | = | 9 | yds. Montreal. |
| 16 | " Montreal | = | 24 | " Paris. |
| 4 | " Paris | = | x | " Toronto. |

Since the left-hand numbers constitute a dividend and the right-hand numbers a divisor, we may cancel factors that are common. Merely writing the numbers and doing this we have—

SAME CANCELLED.

$$\begin{array}{r} 7 = 9 \\ 4 \ 16 = 24 \\ 6 \ 54 = x = 4 \times 7 = 28 \text{ yds. } \textit{Ans.} \end{array}$$

From the preceding principles and illustrations we deduce the following:

RULE FOR CONJOINED PROPORTION.

Write the equivalent terms, as they occur, right and left of the sign of equality, taking care that terms of the same name shall always be on opposite sides.

Multiply all the terms on the same side as the odd term for a dividend and all on the other side for a divisor. The quotient will be the required term.

EXAMPLE 2.—If 25 sheep eat as much hay as 19 goats, and 33 goats as much as 10 cows, and 38 cows as much as 22 horses, how many horses will eat as much as 60 sheep?

| | | | |
|-----------------------|---|--|------------------------------------|
| STATEMENT. | } | Or writing the numbers merely, cancelling and applying the rule. | SAME CANCELLED. |
| 25 sheep = 19 goats | | | $\frac{5}{3} \frac{25}{19} = 19_2$ |
| 33 goats = 10 cows | | | $\frac{3}{2} \frac{33}{10} = 10_4$ |
| 38 cows = 22 horses | | | $\frac{2}{1} \frac{38}{22} = 22_4$ |
| x horses = 60 sheep | | | $x = 60_4$ |

Ans. $4 \times 2 = 8$ horses.

Here, since the term 25 sheep is on the left hand-side, we put the odd term, 60 sheep, on the right-hand side.

NOTE.—The sign = in such questions, merely means equal in value, or equal in time, or equal in effect, &c.

EXAMPLE 3.—If 19 lbs. of tea in Guelph cost as much as 20 lbs. in Hamilton, and 7 in Hamilton as much as $9\frac{1}{2}$ lbs. in Quebec, and 30 lbs. in Quebec as much as $29\frac{1}{2}$ lbs. in Boston, and $8\frac{1}{2}$ lbs. in Boston as much as $5\frac{1}{2}$ lbs. in London, and 10 lbs. in London as much as 57 lbs. in Hong Kong; how many lbs. in Hong Kong are worth 100 lbs. in Guelph?

| | | | |
|---|-----------|-----------------|--|
| STATEMENT. | } | SAME CANCELLED. | |
| 19 Guelph = 20 Hamilton | | | $\frac{19}{20} = 20$ |
| 7 Hamilton = $9\frac{1}{2}$ Quebec | | | $\frac{7}{9\frac{1}{2}} = 9\frac{1}{2}$ |
| 30 Quebec = $29\frac{1}{2}$ Boston | | | $\frac{30}{29\frac{1}{2}} = 29\frac{1}{2} \frac{4}{4}$ |
| $8\frac{1}{2}$ Boston = $5\frac{1}{2}$ London | | | $\frac{8\frac{1}{2}}{5\frac{1}{2}} = 5\frac{1}{2} \frac{10}{10}$ |
| 10 London = 57 Hong Kong | | | $\frac{10}{57} = 57 \frac{10}{10}$ |
| x Hong Kong = 100 Guelph | $x = 100$ | | |

Ans. $10 \times 9\frac{1}{2} \times 5\frac{1}{2} = 506\frac{3}{4}$ lbs.

EXERCISE.

4. If 17 cords of wood are equivalent to 116 lbs. of tea, and 87 lbs. of tea to 23 barrels of flour, and 19 barrels of flour to 34 days' work, and 92 days' work to 57 baskets of peaches, and 31 baskets of peaches to 24 dollars, and 12 dollars to 2 tons of coal; how many cords of wood may be purchased for 35 tons of coal? *Ans.* 135 $\frac{5}{8}$.
5. If 6 lbs. of tea are worth 29 lbs. of sugar, and 17 lbs. of sugar pay for 1 bushel of wheat, and 27 bushels of wheat are equivalent to 4 tons of coal, and 34 tons of coal purchase 15 cows, and 29 cows cost \$1160; how many pounds of tea can be purchased for \$20? *Ans.* 26 $\frac{2}{3}$.

6. If 11 bushels of barley pay for 21 bushels of potatoes, and 19 bushels of potatoes for 29 bushels of oats, and 115 bushels of oats for 44 bushels of wheat, and $14\frac{1}{2}$ bushels of wheat for 38 bushels of peas, and 60 bushels of peas for 55 bushels of rye, and 75 bushels of rye for $11\frac{1}{2}$ bushels of clover seed; for how many bushels of barley will 36 bushels of clover seed pay? *Ans.* $87\frac{1}{4}$.
7. If 16 baskets of pears pay for 29 turkeys, and 17 turkeys for 7 days' work, and $7\frac{1}{4}$ days' work for 187 loaves of bread, and $3\frac{1}{2}$ loaves of bread cost as much as 4 lbs. of veal, and veal is 11 cents per pound, and \$7.92 pay for 63 lbs. of sugar; how many pounds of sugar will 21 baskets of pears purchase? *Ans.* 404 $\frac{1}{2}$.
8. Suppose A can do as much work in 7 days as B can in 11 days, and B as much in 5 days as C can in 8 days, and C as much in 15 days as D can in 21 days, and D as much in 11 days as E can in 5 days; in how many days would A do as much work as E can do in 42 days? *Ans.* 26 $\frac{1}{2}$.
9. If 7 barrels of flour pay for 23 cords of wood, and 6 cords of wood pay for 11 cwt. of beef, and 46 cwt. of beef cost £28, and £77 pay for 9 sheep, and 5 sheep are worth as much as 8 tons of coal; how many barrels of flour may be purchased for 9 tons of coal? *Ans.* 13 $\frac{1}{2}$.
10. If 15s. in N. England be the same in value as 20s. in N. York, and 24s. in N. York the same as 22s. 6d. in N. Jersey, and 30s. in N. Jersey the same as 20s. in Canada; how many pounds in N. England are the same in value as £240 7s. 6d. in Canada? *Ans.* £288 9s.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers following the questions refer to the numbered articles of the section.

1. In how many ways may one number be compared with another with respect to magnitude? (1)
2. What is ratio? (2)
3. What is the difference between the Geometrical and the Arithmetical ratio of numbers? (3)
4. How many ways have we of expressing the ratio of one number to another? (4)
5. Between what kind of quantities only can ratio exist? (5)
6. When are quantities said to be of the same kind? (6)
7. What is a couplet? (7)
8. What is the antecedent?—the consequent? (8)
9. How many kinds of ratio are there? (9)
10. What is a direct ratio? (10)
11. What is an inverse ratio? (11)
12. What is the reciprocal of a quantity? (12)
13. What is a reciprocal ratio? (13)
14. How is the reciprocal ratio of two numbers expressed? (14)
15. Show that "reciprocal ratio" and "inverse ratio" are interchangeable terms? (13)

16. What is a simple ratio? (15)
17. What is a compound ratio? (16)
18. Since a compound ratio does not differ in nature from a simple ratio, why is the term used? (17)
19. How are ratios compounded together? (18)
20. How does multiplying the antecedent or dividing the consequent of a couplet by any number, affect the ratio? (19)
21. How does dividing the antecedent or multiplying the consequent of a couplet by any number, affect the ratio? Why? (19)
22. How does multiplying or dividing both antecedent and consequent of a couplet by any number, affect the ratio? Why? (19)
23. How does it happen that we may cancel any factors common to an antecedent and a consequent, before compounding ratios together? (20)
24. When is a ratio called a *ratio of equality*? (21)
25. When is a ratio called a *ratio of greater inequality*? (21)
26. When is a ratio called a *ratio of less inequality*? (21)
27. How are ratios compared with one another? (22)
28. When equal ratios are added together, what is the nature of the resulting ratio? (24)
29. What effect has adding the same number to both terms of a ratio? (25 and 26)
30. What is Proportion? (27)
31. What are the terms of the two equal ratios called? (28)
32. How many ways are there of expressing Proportion? (29)
33. What is the supposed derivation of the sign ::? (29—Note)
34. How many *terms* must there be in every proportion? (30)
35. When three *numbers* constitute a proportion, what is the repeated term called?—What is the last term called? (31)
36. Point out the distinctions between ratio and proportion. (32)
37. What are "*extremes*" and "*means*"? (33)
38. Prove that if four quantities are proportional, the product of the extremes is equal to the product of the means. (34)
39. What is the test of geometrical ratio? (35)
40. Deduce from this principle a rule for finding any one of the terms when the other three are given. (36.)
41. If $r:w :: x:y$, what does the proportion become? 1st, by composition, 2nd, alternately; 3rd, by conversion; 4th, by division; 5th, inversely. (37)
42. What are the different kinds of Proportion? (38)
43. What other names has Simple Proportion?—Why so called? (39)
44. Give the rule for making the statement in Simple Proportion? (40)
45. Give the rule for finding the unknown quantity after the statement is made? (40)
46. Show that we may cancel any factors that are common to the first term and either of the others, before applying the rule? (41)
47. If any of the terms contain fractions, what is done? (42)
48. If the first and second terms are not of the same denomination, what is the rule? (43)
49. What is Compound Proportion? (44)
50. What other name has Compound Proportion? (45)
51. How many ratios are there in Compound Proportion, and how many of them are perfect? (46)
52. In stating a question in Compound Proportion, what do you make the third term? (47)
53. How do you know whether the other ratios are ratios of greater or less inequality? (47)
54. When the statement is made, how is the answer obtained? (47)
55. Show that before applying the rule we may cancel any factors that are common to any of the first terms, and to the second and third terms. (47—Note)

56. What is Conjoined Proportion? (48)
 57. Why is it sometimes called the Chain Rule? (49)
 58. Give the rule for Conjoined Proportion? (50)
 59. In what sense is the sign = taken in these statements? (50)

MISCELLANEOUS EXERCISE.

(On preceding Rules).

1. What is the ratio compounded of the ratios 7 : 8, 17 : 11, 23 : 29, 319 : 119, and 16 : 69?
2. Reduce £119 16s. 6½d. to dollars and cents.
3. How many days are there from 12th March to the 17th of the following February?
4. Compare together the following ratios, and point out which is greatest and which least, 9 : 13, 21 : 27, 7 : 10, and 11 : 15.
5. From 76·23478 take 19·1342291.
6. Multiply 71324t *undenary* by 23421 *quinary* and divide the result by 14e7 *duodenary*. Give the answer in each scale.
7. If 5·63 cubic inches of water weigh 3·254 ounces avoirdupois, what will be the weight of 7·9 cubic inches of nitric acid having a specific gravity of 1·220?
8. Divide 63 yds. 3 qrs. 2 na. 1 in. of ribbon equally among 17 persons.
9. What is the value of ·913625 of an acre at 67 cents per sq. yard?
10. Multiply ½ of ⅓ of ⅞ of 20 bushels by ·5 × ·6 × ⅞.
11. Of the ratios 6 : 7, 17 : 8, 23 : 11, and 88 : 176, point out (1) which is the greatest, (2) which is the least, (3) which are ratios of greater inequality, (4) which are ratios of less inequality, (5) what is the ratio compounded of these ratios.
12. The population in Canada in 1851 was 1842265, and in 1857 it was estimated at 2571437. What was the rate per cent. of increase?
13. From one-half of two-thirds of eighteen twenty-ninths subtract one-eighth of two-thirds of five-sevenths.
14. Deduct 7 per cent. from 11 feet.
15. What is the value of 79 lbs. of tea at £·00163 per ounce?
16. If 3 men in 2½ days, working 12 hours a day, can cradle a field of wheat containing 20 acres, in how many days can 4 men, working 10 hours a day, cradle a field of wheat containing 35 acres?
17. Find the value of (⅓ of ⅞ × ·02 × 456) ÷ (1⅞ of ⅓ of ½ of 51.)
18. A certain number is divided by 5, the result is divided by ⅞, this result by ⅓, and this last result by ⅞. The last quotient is 2; what was the original number?

19. If 50 barrels of flour in Toronto are worth 125 yards of cloth in New York, and 80 yards of cloth in New York 6 bales of cotton in Charleston, and 13 bales of cotton in Charleston $3\frac{1}{2}$ hogsheads of sugar in New Orleans; how many hogsheads of sugar in New Orleans are worth 1000 barrels of flour in Toronto?
20. Multiply 73.47 by .0063, and divide the result by 17.2345.
21. Reduce 2 roods 7 per. 4 yds. 3 ft. 117 in. to the decimal of 7 acres.
22. Deduct .73 of 11 furlongs from $\frac{2}{3}$ of $\frac{1}{4}$ of $\frac{1}{2}$ of 70 miles.
23. From 274312 *nonary* take 1101011010 *binary*, and multiply the result by 5555 *septenary*. Give the answer in all three scales.
24. Find the l. c. m. of 44, 275, 18, 190, 209, and 225.
25. If 60 men in 6 weeks of 5 working days, of 10 hours each, build an embankment 800 yards in length, 18 feet in mean breadth and 11 ft. in mean height, how many men will make an embankment 8742 feet long, 20 feet wide and 8 ft. high, in 10 weeks, of 6 days each, and 11 working hours to each day?
26. How many divisors has the number 172000?
27. Multiply 42.7 by 9.7123.
28. Deduct 27 per cent. from \$73.42.
29. What are all the divisors of 6300?
30. If $\frac{2}{3}$ of $\frac{3}{4}$ of 34 lbs. of coffee cost $\frac{2}{3}$ of $\frac{2}{3}$ of $\frac{1}{2}$ of a dollar, what will $\frac{3}{4}$ of .7 of .6 of $\frac{2}{3}$ of 90 lbs. cost?
31. If \$2739.18 be divided among 7 men, 2 women, and 11 children, so that each child shall have $\frac{2}{3}$ of a woman's share, and each woman $\frac{3}{4}$ of a man's share, what will be the amount received by each?
32. What is the reciprocal ratio of $\frac{2}{3} : \frac{11}{12}$; the direct ratio of 93 : 17; and the inverse ratio of $\frac{2}{3}$ of $\frac{1}{2}$?
33. Add together $\frac{1}{3}$ of $6\frac{1}{2}$ yards, $\frac{2}{3}$ of $\frac{1}{4}$ of $8\frac{3}{4}$ ft., and $\frac{2}{3}$ of $\frac{1}{3}$ of $7\frac{7}{10}$ inches.
34. What is the ratio compounded of 23 : 7, 4 : 11, 6 : 5, 13 : $11\frac{1}{2}$, and $38\frac{1}{2} : 3$?
35. A pint contains 9000 grains of barley, and each grain is one third of an inch long. How far would the grains in 23 bush. 2 pks. 1 gal. 1 qt. 1 pt. reach if placed one after another?
36. Reduce $\frac{111885}{10385}$ to its lowest terms.
37. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$ in the *octenary* scale.
38. If 17 sheep eat as much grass as 6 cows, and 26 cows require $27\frac{1}{2}$ acres, and 12 acres supply 13 horses, and 11 horses eat as much as 28 goats, how many goats will eat as much as 68 sheep?

39. Suppose that 50 men, by working 5 hours each day, can dig, in 54 days, 24 cellars, which are each 36 feet long, 21 feet wide, and 10 feet deep, how many men would be required to dig, in 27 days, 18 cellars, which are each 48 feet long, 28 feet wide, and 9 feet deep, provided they work only 3 hours each day?

SECTION VI.

PRACTICE.

1. Practice is so called from its being the method of calculation *practised* by mercantile men; it is an abridged mode of performing processes dependent on the Rule of Three—particularly when one of the terms is unity.

The *statement* of a question in practice, in *general terms*, would be—*One quantity of goods: another quantity of goods:: price of former: price of latter.*

2. The simplification of the Rule of Three by means of *practice*, is principally effected, either by dividing the given *quantity* into “parts,” and finding the sum of the prices of these parts; or by dividing the *price* into “parts,” and finding the sum of the prices of each of these parts; in either case, as is evident, we obtain the required price.

3. An Aliquot Part is an exact or even part.

Thus, 2 shillings is an aliquot part of a pound; $12\frac{1}{2}$ cents is an aliquot part of a dollar; 6 months, 4 months, 3 months, 2 months, $1\frac{1}{2}$ months are aliquot parts of a year, &c.

TABLE OF ALIQUOT PARTS.

| Parts of \$1. | Parts of a year. | Parts of a month. | Parts of £1. | Parts of 1s. | Parts of a cwt.* of 112 lbs. |
|----------------------------------|---------------------------------|---------------------------------|------------------------|-----------------------------------|-------------------------------------|
| 50 cts. = $\frac{1}{2}$ | 6 m'ths = $\frac{1}{2}$ | 15 days = $\frac{1}{2}$ | 10s = $\frac{1}{2}$ | 6d = $\frac{1}{2}$ | 56 lb = $\frac{1}{2}$ |
| 33 $\frac{1}{3}$ = $\frac{1}{3}$ | 4 = $\frac{1}{3}$ | 10 = $\frac{1}{3}$ | 6s 8d = $\frac{1}{3}$ | 4d = $\frac{1}{3}$ | 28 lb = $\frac{1}{3}$ |
| 25 = $\frac{1}{4}$ | 3 = $\frac{1}{4}$ | 7 $\frac{1}{2}$ = $\frac{1}{4}$ | 5s = $\frac{1}{4}$ | 3d = $\frac{1}{4}$ | 16 lb = $\frac{1}{4}$ |
| 20 = $\frac{1}{5}$ | 2 = $\frac{1}{5}$ | 5 = $\frac{1}{5}$ | 4s = $\frac{1}{5}$ | 2d = $\frac{1}{5}$ | 14 lb = $\frac{1}{5}$ |
| 16 $\frac{2}{3}$ = $\frac{1}{6}$ | 1 $\frac{1}{2}$ = $\frac{1}{6}$ | 3 = $\frac{1}{6}$ | 3s 4d = $\frac{1}{6}$ | 1 $\frac{1}{2}$ d = $\frac{1}{6}$ | 8 lb = $\frac{1}{6}$ |
| 12 $\frac{1}{2}$ = $\frac{1}{8}$ | 1 = $\frac{1}{8}$ | 2 = $\frac{1}{8}$ | 2s 6d = $\frac{1}{8}$ | 1d = $\frac{1}{8}$ | 7 lb = $\frac{1}{8}$ |
| 8 $\frac{1}{2}$ = $\frac{1}{12}$ | $\frac{1}{2}$ = $\frac{1}{12}$ | 1 = $\frac{1}{12}$ | 2s = $\frac{1}{12}$ | | |
| 6 $\frac{1}{2}$ = $\frac{1}{16}$ | | | 1s 8d = $\frac{1}{16}$ | | |
| 5 = $\frac{1}{20}$ | | | 1s 4d = $\frac{1}{20}$ | | |
| 4 = $\frac{1}{25}$ | | | 1s 3d = $\frac{1}{25}$ | | |
| 2 = $\frac{1}{50}$ | | | 1s = $\frac{1}{50}$ | | |
| | | | | | parts of a qr. of 28 lbs. |
| | | | | | 14 lb = $\frac{1}{2}$ |
| | | | | | 7 lb = $\frac{1}{4}$ |
| | | | | | 3 $\frac{1}{2}$ lb = $\frac{1}{8}$ |
| | | | | | 1 $\frac{1}{2}$ lb = $\frac{1}{16}$ |

* Although we allow but 100 lbs. to the cwt. in Canada, it is often necessary to make calculations with the old cwt., of 112 lbs. This arises from the

EXAMPLE 1.—Find the price of 2783 yards of silk at \$3.37½ per yard.

OPERATION.

| | | | |
|--------|---|------------|--|
| 25 c. | ½ | 2783 | The cost of 2783 yards at \$3.37½ = cost at \$3 + cost at |
| | | 3 | 37½ cents. |
| | | — | 2783 yds. at \$3 comes to 3 times as much as at \$1; i.e., |
| | | 8349 | to 3 times \$2783, or \$8349. 37½ cts. equals 25 cts. + 12½ |
| 12½ c. | ½ | 695.75 | cents, hence, 2783 yds. at 37½ cents = price at 25 cents + |
| | | 347.87½ | price at 12½ cents. |
| | | — | Since 2783 yards at \$1 comes to \$2783, and 25 cents = ¼ |
| | | \$6392.62½ | of a dollar; 2783 yards at 25 cents come to ¼ of \$2783, |
| | | | i.e., to \$695.75. Again, because 2783 yards at 25 cents |
| | | | come to \$695.75 and 12½ cents equals ½ of 25 cents, 2783 yards at 12½ cents |
| | | | will come to ½ of \$695.75; i.e., to \$347.87½. |
| | | | Then 2783 yards at \$3.37½ = price at \$3 + price at 25 cents + price at 12½ |
| | | | cents = \$8349 + \$695.75 + \$347.87½ = \$9392.62½. |

EXAMPLE 2.—What is the cost of 972 oz. of gold dust at £3 14s. 8½d. per oz.?

OPERATION.

| | | | |
|---------|---|------------|--------------------|
| 10s. | ½ | 972 | |
| | | 3 | |
| | | — | |
| | | £2916 | = cost at £3 0 0 |
| 3s. 4d. | ½ | 486 | = cost at 0 10 0 |
| 10d. | ½ | 162 | = cost at 0 3 4 |
| 5d. | ½ | 40 10s. | = cost at 0 0 10 |
| 1½d. | ½ | 20 5 | = cost at 0 0 5 |
| | | 5 1 3d. | = cost at 0 0 1½ |
| | | — | |
| | | £3629 16 3 | = cost at £3 14 8½ |

EXAMPLE 3.—Find the price of 729 days work at £1 7s. 1½d per day.

OPERATION.

| | | | |
|---------|---|-------------|--------------------|
| 5s. | ½ | £729 0 0 | = price at £1 0 0 |
| 1s. 8d. | ½ | 182 5 0 | = price at 0 5 0 |
| 5d. | ½ | 60 15 0 | = price at 0 1 8 |
| ½d. | ½ | 15 3 9 | = price at 0 0 5 |
| | | — | |
| | | 15 3½ | = price at 0 0 0½ |
| | | — | |
| | | £987 18 11½ | = price at £1 7 1½ |

EXAMPLE 4.—What is the cost of 624 bush. 1 pk. 1 gal. 3 qt. of wheat at \$2.87½ per bushel?

OPERATION.

| | | | |
|----------|---|--------|---------------------------------|
| 50 cts. | ½ | 624 | |
| | | 2 | |
| | | — | |
| | | \$1248 | = price of 632 bush. at \$2.00 |
| 25 cts. | ½ | 312 | = price " " at 50 |
| 12½ cts. | ½ | 156 | = price " " at 25 |
| | | 78 | = price " " at 12½ |
| | | — | |
| | | \$1794 | = price of 624 bush. at \$2.87½ |

fact that the latter is still in common use in Great Britain, several of the States of the American Union, &c. The aliquot parts of the new cwt., of 100 lbs. are the same as the aliquot parts of \$1.

$$\begin{array}{r|l}
 1 \text{ pk. } \frac{1}{4} & \$2'87\frac{1}{2} = \text{price of 1 bush.} \\
 1 \text{ gal. } \frac{1}{2} & '71\frac{7}{8} = \text{price of 1 pk.} \\
 2 \text{ qt. } \frac{1}{2} & '35\frac{1}{8} = \text{price of 1 gal.} \\
 1 \text{ qt. } \frac{1}{4} & '17\frac{3}{8} = \text{price of 2 qt.} \\
 & '8\frac{3}{4} = \text{price of 1 qt.}
 \end{array}$$

$$\$1'34\frac{3}{4} = \text{price of 1 pk. 1 gal. 3 qt.}$$

$$\text{Then } \$1794 = \text{price of 624 bushels at } \$2'87\frac{1}{2} \text{ per bushel.}$$

$$1'34\frac{3}{4} = \text{price of 1 pk. 1 gal. 3 qt. at } \$2'87\frac{1}{2} \text{ per bush.}$$

$$\$1795'34\frac{3}{4} = \text{price of 624 bush. 1 pk. 1 gal. 3 qt. at } \$2'87\frac{1}{2} \text{ per bush.}$$

EXAMPLE 5.—What is the price of 96 acres 1 rood $1\frac{1}{2}$ per. at £7 11s. 5½d. per acre?

$$\begin{array}{r|l}
 10\text{s. } \frac{1}{2} & 96 \\
 & \underline{7} \\
 & £672 \text{ 0} = \text{price of 96 acres at } £7 \text{ 0 0} \\
 1\text{s. } 3\text{d. } \frac{1}{4} & 48 \text{ 0} = \text{ " " " at } 0 \text{ 10 0} \\
 1\frac{1}{2}\text{d. } \frac{1}{10} & 6 \text{ 0} = \text{ " " " at } 0 \text{ 1 3} \\
 \frac{1}{2}\text{d. } \frac{1}{2} & 12 = \text{ " " " at } 0 \text{ 0 } 1\frac{1}{2} \\
 & 6 = \text{ " " " at } 0 \text{ 0 } 0\frac{3}{4}
 \end{array}$$

$$£726 \text{ 18} = \text{price of 96 acres at } £7 \text{ 11 } 5\frac{1}{2}$$

$$1 \text{ rood } \frac{1}{4} \quad | \quad £7 \text{ 11 } 5\frac{1}{2}$$

$$\begin{array}{r|l}
 10 \text{ per. } \frac{1}{4} & 1 \text{ 17 } 10\frac{1}{2} + \frac{1}{4} = \text{price of 1 rood.} \\
 4 \text{ per. } \frac{1}{10} & 9 \text{ 5}\frac{1}{2} + \frac{5}{10} = \text{price of 10 perches.} \\
 \frac{1}{2} \text{ per. } \frac{1}{2} & 3 \text{ 9}\frac{1}{2} + \frac{3}{2} = \text{price of 4 perches.} \\
 & 5\frac{1}{2} + \frac{2}{2} = \text{price of } \frac{1}{2} \text{ perch.}
 \end{array}$$

$$£2 \text{ 11 } 7 + \frac{1}{320} \text{ f.} = \text{price of 1 rd. } 14\frac{1}{2} \text{ per. at } £7 \text{ 11s. } 5\frac{1}{2} \text{ per ac.}$$

$$£726 \text{ 18} = \text{price of 96 acres.}$$

$$\text{Ans. } £729 \text{ 9s. } 7\text{d.} + \frac{1}{320} \text{ f.} = \text{price of 96 acres 1 rood } 14\frac{1}{2} \text{ per.}$$

EXAMPLE 6.—What is the cost of $964\frac{1}{6}$ square yards of plastering at 22½ cents per square yard?

$$\begin{array}{r|l}
 20 \text{ cts. } \frac{1}{2} & 964 \\
 2\frac{1}{2} \text{ cts. } \frac{1}{2} & \$192'80 = \text{cost of } 964 \text{ yds. at } 20 \text{ cts.} \\
 & 24'10 = \text{cost of } 964 \text{ yds. at } 2\frac{1}{2} \text{ cts.} \\
 & \underline{\hspace{2cm}} \\
 & \$216'90 = \text{cost of } 964 \text{ yds. at } 22\frac{1}{2} \text{ cts.} \\
 & '16\frac{1}{2} = \text{cost of } \frac{1}{6} \text{ of a yd. at } 22\frac{1}{2} \text{ cts.}
 \end{array}$$

$$\text{Ans. } \$217'06\frac{1}{2} = \text{cost of } 964\frac{1}{6} \text{ yds. at } 22\frac{1}{2} \text{ cts. per yd.}$$

EXERCISE.

7. Required the value of 92647 lbs. of tea at 35 cents per lb.
Ans. \$32426'45.
8. What is the cost of 94937 pails at 1s. 5d. each?
Ans. £6724 14s. 1d.

9. What is the worth of 95972 boxes at $7\frac{1}{2}$ cents? *Ans.* \$7197.90.
10. What is the cost of 62 acres at \$28.80 per acre? *Ans.* \$1785.60.
11. Find the price of 2310 lbs. at $32\frac{1}{2}$ cents per lb. *Ans.* \$750.75.
12. Find the price of 2117 bags at $37\frac{1}{2}$ cents each. *Ans.* 793.87 $\frac{1}{2}$.
13. Find the price of 7506 pair of shoes at 1s. 9 $\frac{1}{4}$ d. a pair. *Ans.* £680 4s. 7 $\frac{1}{2}$ d.
14. What is the value of 1217 lbs. of coffee at $17\frac{1}{2}$ cents. per lb? *Ans.* \$212.97 $\frac{1}{2}$.
15. Find the price of 2103 cords of wood at \$3.07 $\frac{1}{2}$ per cord? *Ans.* \$6466.72 $\frac{1}{2}$.
16. What is the cost of 2096 oz. of gold dust at £3 18s. 10 $\frac{1}{2}$ d. per oz.? *Ans.* £8266 2s. 0d.
17. Required the value of 6 oz. 18 dwt. 20 grs. of silver at \$1.55 per oz.? *Ans.* \$10.75 $\frac{3}{4}$.
18. What is the cost of 98 yds. 3 qrs. 1 na. of cloth at £1 15s. per yard? *Ans.* £172 18s. 5 $\frac{1}{2}$ d.
19. What is the rent of 344 acres 3 roods 15 per. at £4 1s. 1d. per acre? *Ans.* £1398 1s. 0 $\frac{1}{2}$ d.
20. What is the price of 5 oz. 6 dwt. 17 grs. of mercury at 5s. 10d. per oz.? *Ans.* £1 11s. 1 $\frac{3}{4}$ d.
21. Find the price of 4 yards 2 qrs. 3 nails of satin at £1 2s. 4d. per yard? *Ans.* £5 4s. 8 $\frac{1}{2}$ d.
22. Find the price of 32 acres 1 rood 14 perches at £1 16s. per acre? *Ans.* £58 4s. 1 $\frac{1}{2}$ d.
23. Find the price of 3 gals. 5 pts. of spirits of wine at 7s. 6d. per gallon? *Ans.* £1 7s. 2 $\frac{1}{2}$ d.
24. How much will 724 bushels of apples come to at \$1.67 $\frac{1}{2}$ per bushel? *Ans.* \$1212.70.
25. What is the cost of 721 bush. of wheat at \$1.93 $\frac{1}{4}$ per bush? *Ans.* \$1396.93 $\frac{1}{4}$.
26. What is the cost of 4514 rods of fencing at £2 17s. 7 $\frac{1}{2}$ d. per rod? *Ans.* £13005 19s. 3d.
27. What is the price of 3749 $\frac{3}{4}$ acres at £3 15s. 6d. per acre? *Ans.* £14153 17s. 9 $\frac{1}{2}$ d.
- Allowing 112 lbs. to the cwt., find the value of—
28. 17 cwt. 1 qr. 17 lbs. at £1 4s. 9d. per cwt. *Ans.* £21 10s. 8 $\frac{3}{4}$ d.
29. 78 cwt. 3 qrs. 12 lbs. at \$11.55 per cwt. *Ans.* \$910.80.
30. 20 tons 19 cwt. 3 qrs. 27 $\frac{1}{2}$ lbs. at £10 10s. per ton. *Ans.* £220 9s. 11 $\frac{1}{2}$ d. nearly.
31. 219 tons 16 cwt. 3 qrs. at \$45.50 per ton. *Ans.* \$10002.60 $\frac{1}{2}$.

BILLS OF PARCELS.

(No. 1.)

QUEBEC, 16th April, 1859.

Mr. JOHN DAY,

Bought of RICHARD JONES.

| | s. | d. | per yard, | £ | s. | d. |
|--------------------------------------|----|----|-----------|----|----|----|
| 15 yards of fine broadcloth, at..... | 13 | 6 | | 10 | 2 | 6 |
| 24 yards of superfine ditto, at..... | 18 | 9 | " | 22 | 10 | 0 |
| 27 yards of yard wide ditto, at..... | 8 | 4 | " | 11 | 5 | 0 |
| 16 yards of drugget, at..... | 6 | 3 | " | 5 | 0 | 0 |
| 12 yards of serge, at..... | 2 | 10 | " | 1 | 14 | 0 |
| 32 yards of shalloon, at..... | 1 | 8 | " | 2 | 13 | 4 |

Ans. £53 4 10

(No. 2.)

MONTREAL, 24th June, 1859.

Mr. JAMES PAUL,

Bought of THOMAS NORTON.

| | | | |
|--------------------------------------|----|----|-----------|
| 9 pair of worsted stockings, at.... | 4 | 6 | per pair, |
| 6 pair of silk ditto, at..... | 15 | 9 | " |
| 17 pair of thread ditto, at..... | 5 | 4 | " |
| 23 pair of cotton ditto, at..... | 4 | 10 | " |
| 14 pair of yarn ditto, at..... | 2 | 4 | " |
| 18 pair of women's silk gloves, at.. | 4 | 2 | " |
| 19 yards of flannel, at..... | 1 | 7½ | per yard, |

Ans. £23 15 4½

(No. 3.)

TORONTO, 10th July, 1859.

Mr. WM. FILBERT,

Bought of GEORGE PRICE.

| | | |
|------------------------------|-----|----------------|
| 75½ lbs. of sugar, at..... | 7½ | cents per lb., |
| 63 lbs. of tea, at..... | 93 | " |
| 126 lbs. of butter, at..... | 13 | " |
| 35½ lbs. of raisins, at..... | 18½ | " |
| 17 lbs. of sago, at..... | 15 | " |
| 23 lbs. of rice, at..... | 9 | " |
| 58½ lbs. of starch, at..... | 22 | " |

Ans. \$105-02½.

(No. 4.)

HAMILTON, 12th August, 1859.

MR. JOHN JAMES,

Bought of JAMES THOMAS.

| | \$ | cts. |
|---|----|------|
| 198 Sangster's National Arithmetic, at..... | 0 | 60 |
| 197 Robertson's Philosophy of Grammar, at..... | 0 | 50 |
| 83 Hodgins' Geography, at..... | 1 | 00 |
| 57 Sangster's Algebraic Formula, at..... | 0 | 12½ |
| 217 Strachan's Canadian Penmanship, at..... | 0 | 37½ |
| 143 Hodgins' Geography of British Provinces, at.. | 0 | 45 |
| 227 Sangster's Elementary Arithmetic, at..... | 0 | 30 |

Ans. \$521·25

(No. 5.)

NIAGARA, 17th September, 1859.

MR. ALEX. LEITH,

Bought of LAWRENCE MERCER.

| | s. | d. |
|-------------------------------------|----|-------------|
| 9½ yards of silk, at..... | 12 | 9 per yard, |
| 13 yards of flowered ditto, at..... | 15 | 6 " |
| 11½ yards of lustring, at..... | 6 | 10 " |
| 14 yards of brocade, at..... | 11 | 3 " |
| 12½ yards of satin, at..... | 10 | 8 " |
| 11½ yards of velvet, at..... | 18 | 0 " |

Ans. £44 15 10

(No. 6)

KINGSTON, 11th July, 1859.

DR. ALEX. HAMILTON,

Bought of TIMOTHY PESTLE.

| | |
|-----------------------------|--------|
| 14 oz. ipecacuanha, at..... | \$0·67 |
| 23 " laudanum, at..... | 0·89 |
| 17 " emetic tartar, at..... | 1·25 |
| 25 " cantharides, at..... | 2·17 |
| 27 " gum mastic, at..... | 0·61 |
| 56 " gum camphor, at..... | 0·27 |

Ans. \$136·94

(No. 7.)

LONDON, C. W., 1st May, 1859.

MR. JAS. GREY,

Bought of MICHAEL LEWIS.

| | s. | d. | |
|--|----|----|----------------------|
| 15½ lbs. of currants, at..... | 0 | 4 | per lb., |
| 17½ lbs. of Malaga raisins, at..... | 0 | 5½ | " |
| 19¾ lbs. of sun raisins, at..... | 0 | 6 | " |
| 17 lbs. of rice, at..... | 0 | 3½ | " |
| 8½ lbs. of pepper, at..... | 1 | 6 | " |
| 3 loaves of sugar, weight 32½ lbs., at.. | 0 | 8½ | " |
| 13 oz. of cloves, at..... | 0 | 9 | per oz. |
| | | | |
| | | | <i>Ans.</i> £3 13 5½ |

TARE AND TRET.

4. *Tare* and *Tret* is the name given to a rule by means of which merchants calculate the amount of certain allowances which were formerly made in buying and selling goods by weight in large quantities. They were as follows:

1. *Tret*, an allowance for waste in weighing.
2. *Tare*, an allowance for the actual or supposed weight of the *box, bag, barrel, &c.*, containing the goods. And
3. *Cloff*, an allowance of 2 lbs. in every 336 for the turn of the scale in retailing goods.

Of these the only one known in Canada is *Tare*; and as this is always set down in full in the invoice, *Tare* and *Tret*, as a rule, has no existence in Canadian mercantile transactions, and has therefore been altogether omitted.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers after the questions refer to the articles of the section.

1. What is *Practice*? (1)
2. Why is it so called? (1)
3. Of what rule is *Practice* merely a modification? (1)
4. What would be the general statement of a question in *Practice*? (1)
5. How is the process for finding the price of a number of articles simplified by *Practice*? (2)
6. What is an aliquot part? (3)
7. What are the aliquot parts of a dollar? (3)

8. What are the aliquot parts of a year? (3)
9. What are the aliquot parts of a month? (3)
10. What are the aliquot parts of a £? (3)
11. What are the aliquot parts of a shilling? (3)
12. What are the aliquot parts of a cwt. (112 lbs.)? (3)

MISCELLANEOUS EXERCISE.

(On preceding Rules.)

1. Take the number 70204, and, by removing the decimal point (1) multiply it by 100000; (2) divide it by 10000; (3) make it *thousandths*; make it *tenths of billionths*; (5) make it *tenths*; and (6) make it *hundredths of billionths*.
2. Divide 427·1 by ·0000637.
3. What will 19 tons 19 cwt. 3 qrs. 27½ lbs. of hops cost, at £19 19s. 11¼d. per ton?
4. Add together 73·723̄, 11·342̄, 16·713̄, 19·034̄, 713·213437̄, and 12·345678̄.
5. Of the ratios 5 : 7, 9 : 13, 12 : 17, and 7 : 10, point out (1) which is greatest, (2) which is least, (3) what is the ratio compounded of these?
6. If 1 acre of land cost \$80·50, what will 25 acres, 2 rods, 35 rods cost?
7. What is the G. C. M. of 144, 485, and 63.
8. What is the price of 7439 cords of wood at \$3·68½ a cord?
9. Reduce $\frac{134475}{22210}$, $\frac{744335}{44335}$, $\frac{100375}{20000}$, and $\frac{3939}{333}$ to their lowest terms.
10. If 34½ bushels of turnips are worth 17 bushels of potatoes, and 9 bushels of potatoes 59½ lbs. of tea, and 6 lbs. of tea 11½ stone of flour, and 13 stone of flour \$3·60, and 38 cents pay for 12 lbs. of bread; how many bushels of turnips are worth 119 lbs. of bread?
11. If 27 men in 7 days, working 8 hours a day, paint 42 floors, each 20 feet long and 16 feet wide, with 3 coats of paint to each; in how many days, of 11 hours each, will 54 men paint 77 floors, each 24 feet long and 22 feet wide, giving each 5 coats of paint?
12. Take the number 7449164 and by removing the decimal point, make it (1) One hundred thousand times greater.
(2) One million times less.
(3) Hundredths of quadrillionths.
(4) Thousandths.
(5) Tenths of billionths.
(6) Tenths.
13. Reduce 72342 *nonary* to equivalent expressions in the *duodenary*, *senary*, and *ternary* scales, and prove the results by reducing all four numbers to the decimal scale.

14. Express in the decimal scale the greatest and least numbers that can be formed with six digits in the *binary*, *quaternary*, *senary*, *octenary*, and *duodenary* scales.
15. Write down all the divisors of 1728.
16. What is the l. c. m. of the first fifteen *even* numbers, 2, 4, 6, 8, &c. ?
17. From $97\cdot91\dot{3}4\dot{2}$ take $18\cdot123\dot{4}56\dot{7}$.
18. What would be the cost of painting a ceiling 20 ft. 7 in. long and 19 ft. 5 in. 7" wide, at $\$2\cdot87\frac{1}{2}$ per square yard ?
19. Divide 916 acres, 3 roods, 17 per., 7 yards, by 43 acres, 1 rood, 2 per., 17 yds.

SECTION VII.

PERCENTAGE, COMMISSION, BROKERAGE, STOCKS, INSURANCE, CUSTOM-HOUSE BUSINESS, ASSESSMENT.

1. The term Per Cent. is derived from the Latin word *per*, "by" or "for" and *centum*, "a hundred," and means "for a hundred." The term is usually employed to indicate the allowance paid for the use of money, but may also be used to express so much the hundred units of any other quantity.

Thus, the term 5 per cent. on so many dollars, gallons, miles, days, &c., signifies \$5 on every \$100, or 5 gallons on every 100 gallons, or 5 miles on every 100 miles, or 5 days on every 100 days, &c.

2. When the rate per cent. is known, the rate per unit is easily obtained by dividing the rate per cent. by 100.

Thus, 1 per cent. is equal to $\frac{1}{100}$ or '01 per unit.
 2 per cent. is equal to $\frac{2}{100}$ or '02 per unit.
 7 per cent. is equal to $\frac{7}{100}$ or '07 per unit.
 9 per cent. is equal to $\frac{9}{100}$ or '09 per unit.
 10 per cent. is equal to $\frac{10}{100}$ or '10 per unit.
 18 per cent. is equal to $\frac{18}{100}$ or '18 per unit.
 39 per cent. is equal to $\frac{39}{100}$ or '39 per unit.
 95 per cent. is equal to $\frac{95}{100}$ or '95 per unit.
 125 per cent. is equal to $\frac{125}{100}$ or 1'25 per unit.
 378 per cent. is equal to $\frac{378}{100}$ or 3'78 per unit.

- $\frac{1}{2}$ per cent. is equal to $\frac{1}{100}$ or '005 per unit.
 $\frac{3}{4}$ per cent. is equal to $\frac{3}{100}$ or '0025 per unit.
 $\frac{7}{8}$ per cent. is equal to $\frac{7}{100}$ or '0075 per unit.
 $\frac{1}{4}$ per cent. is equal to $\frac{1}{100}$ or '00125 per unit.
 $6\frac{1}{2}$ per cent. is equal to $\frac{6\frac{1}{2}}{100}$ or '065 per unit, &c.

EXERCISE.

1. What rate per unit is equivalent to 1·6 per cent., 11 per cent., 17 per cent., 63 per cent.?
2. What rate per unit is equivalent to 6 per cent., 25 per cent., 137 per cent.?
3. What rate per unit is equivalent to $8\frac{1}{2}$ per cent., $9\frac{1}{4}$ per cent., $2\frac{1}{2}$ per cent.?
4. What rate per unit is equivalent to $\frac{1}{3}$ per cent., $\frac{7}{8}$ per cent., $8\frac{3}{4}$ per cent.?
5. At $6\frac{1}{4}$ per cent., how much is it for 1? *Ans.* '0625.
6. At $18\frac{3}{4}$ per cent., how much is it for 1? *Ans.* '186.
7. At $23\frac{3}{4}$ per cent., how much is it for 1? *Ans.* '23625.
8. At 2·734 per cent., how much is it for 1? *Ans.* '02734.
9. At 82·7 per cent., how much is it for 1? *Ans.* '827.
10. At $19\frac{1}{2}$ per cent., how much is it for 1? *Ans.* '193.

3. To find the percentage of any given number—

RULE.

Multiply the given number by the rate per unit expressed decimally, and point off the product as directed in Art. 53, Sec. II.

EXAMPLE 11.—What is 7 per cent. on \$673·93?

OPERATION.

$$\$673\cdot93 \times '07 = \$47\cdot1751$$

EXPLANATION.—7 per cent. is equivalent to '07 per unit; or, in other words, the percentage on each dollar is 7 cents. It is obvious then that the percentage on the whole sum will be as many times 7 cents as the sum contains dollars; that is $673\cdot93 \times 7$.

EXAMPLE 12.—What is $6\frac{1}{4}$ per cent. on \$2934?

$$\text{Ans. } \$2934 \times '065 = \$190\cdot71.$$

EXAMPLE 13.—What is $47\frac{1}{2}$ per cent. on 7893 gallons of molasses?

$$\text{Ans. } 7893 \text{ gal.} \times '4775 = 3768\cdot9075 \text{ gallons.}$$

EXERCISE.

14. What is 5 per cent. of \$742·10? *Ans.* \$37·10 $\frac{1}{2}$.
15. What is 11 per cent. of \$1000? *Ans.* \$110.
16. How much is 10 per cent. of \$734·10? *Ans.* \$73·419.

17. How much is $87\frac{1}{2}$ per cent. of \$1624.50? *Ans.* \$1421.4375.
18. What is $12\frac{1}{2}$ per cent. on \$994.70? *Ans.* \$124.3375.
19. What is $8\frac{1}{2}$ per cent. on \$777.50? *Ans.* \$68.03 $\frac{1}{2}$.
20. What is $2\frac{1}{2}$ per cent. of \$7135.80? *Ans.* \$160.5555.
21. A merchant imports 2740 boxes of oranges, and finds, upon receiving them, that 20 per cent. of the whole quantity are decayed. To how many boxes was his loss equivalent?
Ans. 548 boxes.
22. A gentleman purchases a farm for \$7490, agreeing to pay 10 per cent. down, 17 per cent. at the end of the first year, 27 per cent. at the end of the second year, and 46 per cent. at the end of the third year. What is the amount of each payment?
Ans. \$749 down.
\$1273.30 at the end of 1st year.
\$2022.30 at the end of 2nd year.
\$3445.40 at the end of 3rd year.
23. What is the difference between $4\frac{1}{2}$ per cent. of \$740 and $2\frac{1}{2}$ per cent. of \$1680? *Ans.* \$8.70.
24. If I purchase 729 gallons of brandy and lose 11 per cent. by leakage, &c., how much have I remaining?
Ans. 648 $\frac{81}{100}$ gallons.
25. Add together 25 per cent. of \$763.22, 16 per cent. of \$847.16, and $6\frac{1}{2}$ per cent. of \$1234.17. *Ans.* \$403.486225.
26. A person dying leaves an estate worth \$17429.40 to be divided among his three sons. The eldest is to receive 43 per cent. of the whole, the second 37 per cent. of the whole, and the youngest son the remainder; what is the share of each?
Ans. The eldest receives \$7494.64 $\frac{1}{2}$, the second \$6448.87 $\frac{1}{2}$, and the youngest \$3485.88.
27. A merchant purchases vinegar to the amount of 68978 gallons and finds, upon receiving it, that 36 per cent. had leaked away. What was his loss? *Ans.* 24832.08 gallons.
28. A brick kiln contains 29800 bricks, and it is found after burning that 17 per cent. of the entire quantity are worthless; how many good bricks were there in the kiln?
Ans. 24734.

COMMISSION.

4. Commission is the percentage charged by agents, or commission merchants, for their services in purchasing or selling goods, collecting bills, &c.

The person who buys or sells goods for another is called an Agent, a Commission Merchant, a Factor, or a Correspondent,

5. To find the commission or any sum at a given rate per cent. is simply to find the percentage on that sum, and the rule employed is the same as that in Art. 3, viz :

Multiply the given amount by the rate per unit expressed decimally.

EXAMPLE 1.—What is the commission on \$790·80 at 3 per cent.?
Ans. $\$790\cdot80 \times \cdot03 = \$23\cdot724$.

EXAMPLE 2.—A commission merchant sells goods to the amount of \$7982·75; what is his commission at $2\frac{3}{4}$ per cent.?
Ans. $\$7982\cdot75 \times \cdot0275 = \$219\cdot525625$.

EXERCISE.

3. What is the commission on \$1000 at $4\frac{1}{2}$ per cent.?
Ans. \$45.
4. What is the commission on \$1678·30 at $2\frac{1}{2}$ per cent?
Ans. \$37·76175.
5. What is the commission on \$7531·19 at $3\frac{1}{2}$ per cent.?
Ans. \$282·419625.
6. Find the commission on \$508·60 at $1\frac{1}{2}$ per cent.?
Ans. \$6·3575.
7. Find the commission on \$7863·50 at $1\frac{1}{2}$ per cent.?
Ans. \$137·61125.
8. An agent collects debts to the amount of \$878·30; what is his commission at $2\frac{1}{2}$ per cent.?
Ans. \$21·9575.
9. A correspondent purchases teas for me to the amount of \$7193·16; what have I to pay him for commission at $3\frac{1}{2}$ per cent.?
Ans. \$224·78625.
10. A commission merchant sells goods to the amount of \$6734·10; what is his commission at 17 per cent.?
Ans. \$1144·797.
11. An agent sells 718 barrels of flour at \$7·13 a barrel; what is his commission at $4\frac{1}{2}$ per cent.?
Ans. \$217·57195.
12. A commission merchant disposes of 8243 bushels of wheat at \$1·85 per bushel; what is the amount of his commission at $5\frac{1}{2}$ per cent.?
Ans. \$857·7871875.

BROKERAGE.

6. Brokerage is the *percentage* charged by money dealers, called *Brokers*, for negotiating *notes*, *mortgages*, *bills of exchange*, &c., or for buying or selling stocks, &c.

7. Brokerage is merely another name for commission, and is computed by the same rule.

EXERCISE.

13. What is the brokerage on \$7893·87 at 2 per cent.?
Ans. \$157·8774.
14. What is the brokerage on \$8000 at $\frac{7}{8}$ per cent.? *Ans.* \$70.
15. What is the brokerage on \$8643·22 at $1\frac{1}{4}$ per cent.?
Ans. \$108·04025.
16. What is the brokerage on \$78963·80 at $\frac{7}{8}$ per cent.?
Ans. \$690·93325.
17. What is the brokerage on \$1987·27 at $3\frac{3}{4}$ per cent.?
Ans. \$74·522625.

8. Commission and Brokerage should both be computed on the amount of money *collected* or *invested*.

For example: If I receive \$10000 to invest and charge 5 per cent., my brokerage would be \$500 if I invested the whole \$10000; but if, as is usually the case, I am requested to deduct, from the amount sent, my brokerage or commission, and invest the remainder, it would obviously be unjust to charge commission on the whole amount,—i. e., on the sum invested and also on the sum I retain for commission. Hence, in all cases, the sum actually expended is the proper basis upon which to compute the commission, brokerage, &c.

9. To compute commission or brokerage when it is to be deducted in advance from a given amount, and the balance invested:—

RULE.

1. Divide the given amount by \$1, plus the commission on \$1, and the result will be the sum to be invested.
2. Subtract the part to be invested from the given amount, and the remainder will be the commission or brokerage.

EXAMPLE 18.—A correspondent receives \$16782, with instructions to deduct his commission at $3\frac{3}{4}$ per cent., and invest the balance in sugar at $9\frac{1}{2}$ cents per pound. How much sugar does he ship to his employer, and what is his commission?

OPERATION,

$$\$16782 \div 1.035 = \$16214.49275 = \text{sum to be invested.}$$

$$\$16782 - \$16214.49275 = \$567.50725 = \text{commission.}$$

$$\$16214.49275 \div 9\frac{1}{2} \text{ cents} = 170678.871 \text{ lbs. } \textit{Ans.}$$

EXPLANATION.—The commission on \$1, at the rate of $3\frac{3}{4}$ per cent., is \$0.035. Hence, for every time he receives \$1.035, he keeps \$0.035 for commission and invests \$1. It is plain, then, that if we divide the given amount, \$16782, by \$1.035, or, in other words, find how often the latter sum is contained in the former, we shall find how often he invests \$1; i. e., how many dollars he invests.

The work may be proved by finding the commission on the sum invested (Art. 5), and comparing it with the commission as found by deducting the sum invested from the whole sum sent. If these are equal, the work is correct.

EXERCISE.

19. An agent receives \$4000, with instructions to purchase Great Western Railway stock. After deducting his brokerage at $1\frac{1}{2}$ per cent., how much money had he to invest and what was his brokerage? *Ans.* Invested \$3950·61728.
Commission \$49·38271.
20. A merchant sends his agent \$7500, with instructions to deduct his commission at $4\frac{1}{2}$ per cent., and purchase laces with the remainder. What is the commission, and what sum was expended in laces? *Ans.* Commission \$322·96651.
Invested \$7177·03349.
21. A commission merchant receives \$8470, with instructions to purchase the best brand of Canadian superfine flour at \$6·40 per barrel. He is to receive out of this sum 5 per cent. on the amount he invests. How many barrels of flour does he purchase? *Ans.* $1260\frac{5}{8}$ barrels.
22. A broker receives \$11000, with instructions to invest it in Bank stock—deducting his brokerage at $\frac{7}{8}$ per cent. What sum had he to invest? *Ans.* \$10904·584882.
23. If I remit to my agent \$13000, instructing him to purchase broad cloth at \$3·63 per yard, and he keeps $4\frac{1}{2}$ per cent. on the sum invested, for commission; how much cloth does he send me, and what is his commission?
Ans. 3427·0499 yds. of cloth.
\$559·8086 commission.

S T O C K .

10. Stock is a term used to denote the *Capital* of moneyed institutions, as Banks, Railroad Companies, Gas Companies, Insurance Companies, Manufactories, &c.

11. Stock is usually divided into portions of \$100 or £100 each, called *shares*, and the different individuals owning these are called *shareholders* or *stockholders*.

12. The *Association of Shareholders*, is called a *Company* or *Corporation*; and the Act of Parliament specifying their corporate powers, rights, and privileges is called a *charter*.

13. The *nominal* or *par* value of a share is its original cost of valuation.

14. The market or real value of a share is the sum for which it can be sold.

15. The rise and fall in the value of stock is reckoned at a certain per cent. on its *nominal* or *par* value.

16. When stocks sell for their original cost or valuation, they are said to be *at par*; when they sell for more than their original valuation, they are said to be *at a premium* or *advance*, or *above par*; when they do not bring their original cost or valuation, they are said to be *at a discount*, or *below par*.

NOTE.—*Par* is a Latin word, and means *equal* or *a state of equality*. Stock is *at par* when a hundred-dollar share sells for \$100; it is *above par* when it brings more than \$100, and *below par* when it will not bring as much as \$100.

17. Persons who deal in stocks are called *stock-brokers* or *stock-jobbers*.

18. To find how much stock either *above* or *below par* a given sum will purchase:—

RULE.

Divide the given amount by the worth of \$1 stock, and the result will be the stock required.

EXAMPLE 1.—How much stock at 10 per cent. *below par* can be purchased for \$25000. *Ans.* $\$25000 \div 0.90 = \$27777.77\frac{1}{2}$.

EXPLANATION.—When stock is 10 per cent. *below par*, each share of \$100 sells for only \$90, i. e. \$90 money will purchase \$100 stock, therefore \$0.90 money will purchase \$1 stock and the given sum will purchase \$1 stock as often as it (the given sum) contains \$0.90.

EXAMPLE 2.—How much stock at 15 per cent. *premium* may be purchased for \$7000? *Ans.* $\$7000 \div 1.15 = \6086.9565 .

EXPLANATION.—When stock is 15 per cent. *above par*, it requires \$115 money to purchase \$100 stock, or \$1.15 money to purchase \$1 stock. Hence if we divide the whole sum to be invested by the value of \$1 stock, it is evident we must get the amount of stock produced.

EXAMPLE 3.—I own \$16400 stock of the Bank of Montreal, and sell out at 13 per cent. *premium*. What do I receive?

Ans. $\$16400 \times 1.13 = \18532 .

EXPLANATION.—Each \$100 stock bring me \$113 money, or \$1 stock brings \$1.13 money, therefore \$16400 stock must bring $\$16400 \times 1.13$ money.

EXERCISE.

- A person has \$9000 which he wishes to invest in Grand Trunk Railway shares, then selling at 17 per cent. *discount*, what amount of stock can he purchase? *Ans.* \$10843.373.
- If I invest \$8500 in Upper Canada Bank stock, which is selling 11 per cent. *above par*, what amount of stock do I receive?
Ans. \$7657.6576.

6. If I remit to my agent \$17500, with instructions to deduct his brokerage at $1\frac{1}{4}$ per cent., and invest the remainder in Great Western Railroad stock, then selling at 7 per cent. premium, what amount of stock do I receive? *Ans.* \$16153·22.
7. If I receive \$20000, with instructions to deduct my commission at $1\frac{1}{4}$ per cent., and invest the balance in stock, which is then selling at 3 per cent. discount, what amount of stock do I remit to my employer? *Ans.* \$20263·937.
8. Mr. A. owns 200 shares in the Canada Life Assurance Company. The par value is \$100 a share, the stock at a premium of $5\frac{1}{2}$ per cent.; if I purchase it through a broker who charges me $\frac{7}{8}$ per cent. for the transaction; how much do my 200 shares cost me? *Ans.* \$21284·625.

INSURANCE.

19. Insurance is a written agreement by which an individual or an incorporated company binds itself, in consideration of a certain sum paid in advance, to exempt the owners of certain kinds of property, as houses, household furniture, merchandise, ships, &c., from loss by fire, shipwreck, or other calamity.

20. The *Written Instrument*, or contract between the parties, is called a *Policy of Insurance*.

21. The sum paid for the insurance is called the *Premium*, and is usually a certain per cent. on the sum for which the property is insured.

22. Houses, merchandise, furniture, &c., are usually insured against risk of fire for the year, or other specified time.

NOTE.—The rate of insurance on dwelling houses, stores, goods, household furniture, &c., varies from $\frac{1}{4}$ to 2 per cent. per annum, on the sum insured, according to the character and position of the tenement; vessels are insured for the voyage or the year.

23. To compute the premium for insurance for 1 year, or a specified time, we use the same rule as for Commission or Brokerage.

EXAMPLE 1.—If I insure my house and furniture for \$7389, at the rate of $1\frac{1}{4}$ per cent. per annum, what premium must I pay yearly? *Ans.* $\$7389 \times \cdot 0125 = \$92\cdot 3625$.

EXPLANATION.— $1\frac{1}{4}$ per cent., i. e. \$1·25 per \$100, is equal to \$0·0125 per dollar. The premium therefore will be as many times \$0·0125 as the sum insured contains \$1 i. e. the premium will be \$0·0125 \times 7389.

EXERCISE.

2. What is the premium for insurance on \$7500, at $1\frac{1}{2}$ per cent. ?
Ans. \$131.25.
3. What is the premium for insurance on \$8375, at $\frac{3}{4}$ per cent. ?
Ans. \$62.8125.
4. What is the premium for insurance on \$6000, at $1\frac{7}{8}$ per cent. ?
Ans. \$112.50.
5. What is the premium for insurance on \$5000, at \$1.17 per cent. (i. e. per \$100) ?
Ans. \$58.50.
6. What is the premium for insurance on \$6400, at \$0.90 per cent. ?
Ans. \$57.60.
7. What is the premium for insurance on \$4500, at \$0.35 per cent. ?
Ans. \$15.75.
8. What premium must I pay for insuring a cargo of flour worth \$36000, from Quebec to Liverpool, at \$3 per cent. ?
Ans. \$1080.
9. A firm owning four steamers running on Lake Ontario, effect an insurance with a company in Toronto on each, to the amount of \$27000, paying \$4.82 per cent. (i. e. $4\frac{82}{100}$ per cent.) What is the total premium on the four steamers ?
Ans. \$5205.60.
10. What is the annual premium on an insurance for \$39000, at $2\frac{1}{2}$ per cent. ?
Ans. \$868.
11. A farmer insures his barns and their contents to the amount of \$17800. What premium does he pay at $\frac{1}{2}$ per cent. ?
Ans. \$89.
12. A vessel running between Hamilton and Oswego is insured for \$12350, at the rate of $1\frac{3}{4}$ per cent. per month. To what does the premium of insurance amount for 7 months, beginning with the 10th of April and ending with the 10th of November ?
Ans. \$1235.

24. To find what sum must be insured on property so that, if destroyed, its value and the premium may both be recovered:—

RULE.

Divide the value of the property by \$1, minus the premium on \$1 at the given rate per cent.

EXAMPLE 13.—A ship-owner wishes to insure a vessel valued at \$17450, so that if it be wrecked he may recover both the value of the vessel and the premium. In order to do so, for what sum must he insure, at \$4.60 per cent. ?

Ans. $\$17450 \div .954 = \$18291.40461.$

EXPLANATION.—If I insure goods to the value of \$100, at 4½ per cent., and they are destroyed, I receive only \$95.40 towards my loss, since I paid \$4.60 for insurance; that is, for every \$1 of my loss I receive \$0.954. Since, then, the recovery of \$0.954 requires \$1 to be insured, the recovery of \$17450 will require as many dollars to be insured as \$0.954 is contained times in \$17450.

PROOF.— $\$18291.40461 \times .046 = \841.40461 = the premium, and $\$18291.40461 - \$841.40461 = \$17450$ = value of the vessel.

EXAMPLE 14.—What sum must be insured on a house valued at \$6000, at 3 per cent., so that in case of fire the value of both premium and property may be secured?

Ans. $\$6000 \div .97 = \6185.567 .

EXPLANATION.—For every dollar I lose (taking premium into account) I receive 97 cents; that is, in order to receive 97 cents, I must insure for \$1, and in order to receive \$6000, without any loss, I must insure for $\$6000 \div .97 = \6185.567 .

EXERCISE.

15. For what sum must I insure a cargo valued at \$17000, so that in case the whole is lost I may recover both the value of the property and the premium of 3½ per cent.?
Ans. \$17616.58.
16. For what sum must I insure on \$22750 in order to cover both the premium of 6 per cent. and the value of the property insured?
Ans. \$24202.127.
17. What sum must be insured at 2½ per cent. on property worth \$15000 so that the owner may be secured against all loss?
Ans. \$15345.2685.
18. A steamer worth \$33000 is insured at 5½ per cent. for such a sum that in case of its becoming a total wreck, the owners recover both the worth of the vessel and the premium of insurance. For what sum is it insured?
Ans. \$35013.2625.

CUSTOM HOUSE BUSINESS.

25. All goods coming into Canada from Foreign countries are required by law to be landed at certain places or ports called *Ports of Entry*.

26. At every Port of Entry in Canada the Government has an establishment called a *Custom House*, with one or more officers attached to it, called *Custom House Officers*.

27. A certain charge called a *Duty*, fixed by Act of Parliament, is made upon nearly all goods entering Canada from Foreign countries.

28. It is the business of the Custom House Officers to inspect the cargoes of all vessels entering at any of these

ports, to examine the invoice of goods, collect the duties, &c., &c.

29. Besides the duties on merchandise, all vessels engaged in commerce are required to pay certain charges for the privilege of entering the port, &c.; these charges are called harbor dues.

30. The duties levied by law on goods imported into Canada are of two kinds:

- 1st. Specific duties.
- 2nd. Ad Valorem duties.

31. A specific duty is a certain sum levied on the ton, cwt., lb., gallon, square yard, &c., of a particular kind of merchandise, as so much per square yard on woollens, flannels or cloths, so much per lb. on tea, so much per gallon on brandy, wine, &c.

32. An ad valorem duty is a certain percentage on the actual cost of the goods in the country in which they were purchased.

Thus an ad valorem duty of 10 per cent. on satin purchased in France is a charge for duty of 10 per cent. of the sum the invoice of satin cost in France.

NOTE 1.—The term *ad valorem* is from the Latin, and means *according to the value*, i. e., *upon the value*.

NOTE 2.—An invoice is a written statement of the goods, showing the quantity of each sort and its value or price.

33. In the United States Custom Houses certain legal allowances are made for draft, tare, leakage, &c., before specific duties are imposed. In Canada, however, as before remarked, (Art. 4., Sect. VI.,) these are not known, the tare being found by actually weighing one or more of the boxes, &c., containing the goods, and the leakage by gauging the cask.

NOTE.—At present (1859) the various kinds of spirits are the only articles upon which specific duties are charged by the Canadian Tariff.

34. To calculate the specific duty on an invoice of goods:—

RULE.

Deduct the tare, leakage, &c., and multiply the remainder by the given duty per gallon, lb., yard, &c.

EXAMPLE 1.—At $4\frac{1}{2}$ cents per lb. what is the specific duty on 7 bags of coffee weighing 73 lbs. each, allowing 4 lbs. per 100 for tare.

OPERATION.

$$73 \times 7 = 511 \text{ lbs.} = \text{gross weight.}$$

$$511 \times .04 = 20\frac{1}{2} \text{ lbs.} = \text{tare.}$$

$$490\frac{1}{2} = \text{net at } 4\frac{1}{4} \text{ cents per lb.} = 490\frac{1}{2} \times 4\frac{1}{4} = \$208463. \text{ Ans.}$$

EXAMPLE 2.—What is the specific duty on 10 chests of tea, the net weight 783 lbs., at 11 cents per lb.?

OPERATION.

$$783 \times 11 = 8613 \text{ cents} = \$86.13. \text{ Ans.}$$

EXERCISE.

3. What is the specific duty, at $3\frac{1}{2}$ cents per lb., on 5 hhd. of sugar, each weighing 1347 lbs., allowing tare 6 lbs. per 100? *Ans.* \$221.58.
4. What is the specific duty, at \$1.20 per 100 lbs., on 11 bags of rice, each weighing 127 lbs., allowing 3 lbs. per 100 for tare? *Ans.* \$16.26.
5. What is the specific duty, at 13 cents per gallon, on 129 gallons of oil? *Ans.* \$16.77.
6. What is the specific duty, at $5\frac{1}{2}$ cents per lb., on 207 drums of figs, each weighing 31 lbs., allowing $2\frac{1}{4}$ lbs. a drum for tare? *Ans.* \$342.1968.
7. What is the specific duty, at 47 cents per yard, on 214 yards of black silk velvet? *Ans.* \$100.58.

35. To find the *ad valorem* duty on an invoice of merchandise:—

RULE.

Multiply the value of the goods at the place in which they were purchased by the per cent. charged, expressed decimally, and the result will be the duty required.

EXAMPLE 8.—What is the *ad valorem* duty, at 27 per cent., on an invoice of brandy which cost \$7493.70?

OPERATION.

$$\$7493.70 \times .27 = \$2023.299. \text{ Ans.}$$

EXAMPLE 9.—What is the *ad valorem* duty, at 19 per cent., on a quantity of broadcloth which cost \$4116.40?

OPERATION.

$$\$4116.40 \times .19 = \$782.116. \text{ Ans.}$$

EXERCISE.

10. What is the *ad valorem* duty, at 21 per cent., on an invoice of silks which cost \$17429.80? *Ans.* \$3660.2580.
11. What is the *ad valorem* duty, at $7\frac{1}{2}$ per cent., on 40 boxes of tea which cost \$2929.16? *Ans.* \$219.012.
12. What is the *ad valorem* duty, at 25 per cent., on an invoice of jewellery which cost \$71342.99? *Ans.* \$17835.725.

13. What is the ad valorem duty, at 20 per cent., on an invoice of boots and shoes which cost \$913.73? *Ans.* \$182.746.
 14. What is the ad valorem duty at 33 per cent., on an invoice of French silks which cost \$14713.19? *Ans.* \$4855.3527.

ASSESSMENT OF TAXES.

36. A tax is a certain sum required to be raised by a municipality for local improvement, payment of officers, and other general purposes. It is collected from each citizen in proportion to the value of his property.

37. In levying taxes the first thing to be done is to make a complete inventory of the value of all the property in the city, town, township, &c., in which the tax is to be raised. This inventory is made by officers called Assessors, appointed by the municipality.

38. To calculate the amount of taxes any one individual has to pay:—

RULE.

Divide the whole sum to be levied by the whole value of rateable property in the town, township, &c.: the quotient will be the sum to be paid on each dollar.

Multiply the rate per dollar by the amount of the person's property, and the product will be the amount of his tax.

EXAMPLE 1.—A certain township requires to raise the sum of \$14729.00 for general purposes; the whole amount of rateable property in the municipality being set down at \$2743500, what proportion must I bear if my property is assessed at \$7490.00.

OPERATION.

$$\begin{aligned} \$14729 \div \$2743500 &= \$0.005368 = \text{rate per dollar.} \\ \$0.005368 \times 7490 &= \$40.20632. \quad \text{Ans.} \end{aligned}$$

EXERCISE.

2. The assessment rolls of a town show the value of the rateable property to be \$7142300. A tax of \$23900 is to be levied for general purposes, how much is my proportion, my property being set down at \$14729.50. *Ans.* \$49.2878.
 3. A tax of \$100000 is to be levied on a county having rateable property to the value of \$5793000, what is the amount borne by A, whose property is valued at \$18600? *Ans.* \$321.0732.
 4. In the last example what would be the amount of B's tax, the value of his property being \$7500? *Ans.* \$129.465.
 5. In the same example what would be the amount of C's tax, his property being assessed at \$11400. *Ans.* \$196.7868.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numerals after the questions refer to the numbered articles of the section.

1. What is the meaning and derivation of the term per cent.? (1)
2. When the rate per cent. is known, how is the rate per unit obtained? (2)
3. How do we ascertain the percentage on any given number? (3)
4. What is commission? (4)
5. What is the person who sells goods for another called? (4)
6. How do we find the commission on any given sum? (5)
7. What is brokerage? (6)
8. How is the brokerage on any sum computed? (7)
9. Upon what sum should commission and brokerage be computed? (8)
10. Explain this by an example.
11. How do we compute commission or brokerage when it is to be deducted in advance from a given amount, and the balance invested? (9)
12. How is this rule proved? (9)
13. What is understood by the term Stock? (10)
14. How is stock usually divided? (11)
15. What is meant by the terms Shareholders, Corporation, and Charter? (11 and 12)
16. What do you understand by the *nominal* or *par* value of Stock? (13)
17. What is meant by the *market* or *real* value of Stock? (14)
18. When is Stock said to be *at par*? when at a *premium* or *above par*? and when at a *discount* or *below par*? (16)
19. What is the meaning of the term *par*? (16, note.)
20. What are persons who deal in Stocks called? (17)
21. When Stock is either *above* or *below par*, how do we find how much of it a given sum will purchase? (18)
22. What is Insurance? (19)
23. What is a Policy of Insurance? (20)
24. What is meant by the *Premium of Insurance*? (21)
25. For what length of time is property *usually* insured? (22)
26. How do we compute the premium of insurance on any amount of goods, property, &c.? (23)
27. How do we compute the amount for which we must insure in order to cover both the value of the property and the premium paid? (24)
28. How may the truth of this rule be proved? (24)
29. What are Ports of Entry? (25)
30. What is the duty of Custom House Officers? (28)
31. What are duties? (27)
32. What are harbor dues? (29)
33. What different kinds of duties are levied on goods in Canada? (30)
34. What are *specific duties*? (31)
35. What is an *ad valorem duty*? (32)
36. What is the meaning of the term *ad valorem*? (32)
37. What is an invoice? (32)
38. What is the rule for computing *specific duties*? (34)
39. What is the rule for calculating *ad valorem duties*? (35)
40. What is a tax? (36)
41. How are taxes imposed? (and 38)

SECTION VIII.

INTEREST, DISCOUNT, EQUATION OF PAYMENTS, AND PARTNERSHIP.

1. Interest is the sum allowed for the use of money, and is usually reckoned at a certain rate per cent. per annum; that is, so many pounds for the use of £100 for one year, so many dollars for the use of \$100 for one year, &c.

NOTE.—The term *per cent.* means *per hundred*; *per annum* means *per year*.

2. Interest differs from Commission, Brokerage, &c., in that the latter are computed at a certain per cent. without regard to time, while interest is calculated at a certain rate per cent. for one year, and consequently for longer and shorter periods in like proportion.

3. The Principal is the sum lent.

4. The Rate *per cent.* is the sum paid for the use of each hundred dollars, pounds, &c.

5. The Rate *per unit* is the sum paid for the use of each dollar, pound, &c.

6. The Interest is the whole sum received for the use of the principal.

7. The Amount is the sum obtained by adding together the principal and the interest.

Thus, if I lend \$200 for a year, on the agreement that I am to receive interest at the rate of 7 per cent. (*per annum*, understood), at the end of the year I receive back the \$200, and in addition \$14 for interest. Here,

\$200'00 is the principal.

7'00 is the rate per cent.

0'07 is the rate per unit.

14'00 is the interest.

214'00 is the amount = principal + interest.

8. Interest is either Simple or Compound.

9. Money is lent at Simple Interest when the interest is not added to the principal so as to bear interest.

Thus, if \$100 be lent at simple interest at 5 per cent., the *principal* remains unchanged, being always \$100, and the *interest* for each successive year is \$5.

10. Money is lent at Compound Interest when the interest, as it falls due from time to time, is added to the principal; the sum thus obtained constituting a *new* principal for the ensuing year, half year, quarter, &c., as the case may be.

Thus, if \$100 be lent at 5 per cent. per annum compound interest, the principal changes at the end of each year; being \$100 for the first year, \$105 (i. e. former principal + its interest) for the second, \$110.25 for the third, &c. The interest is consequently \$5 for the first year, \$5.25 for the second, \$5.5125 for the third, &c.

SIMPLE INTEREST.

11. Questions in Interest are dependent on Proportion, and may all readily be solved by one or more statements in the Rule of Three; but in order to deduce special rules, we shall represent the different quantities by their initial letters, and thus obtain a series of algebraic formulas, which, translated, become the common arithmetical rules for interest.

It is to be presumed that the pupil has made sufficient progress in Algebra before he arrives at this point, to readily understand what follows. The operations involved are of the simplest kind, and may without difficulty be comprehended, even by those wholly ignorant of Algebra. The only part, however, absolutely necessary for working any problem in interest, is the *interpretation* of the *formula*, i. e. the *arithmetical rule*, and this we have always appended. A glance at the formulas and the corresponding rules will show how much less labor is necessary to remember the former than the latter;—and indeed the pupil should be required to deduce from time to time any formula he may find it necessary to use.

NOTE.—When two or more letters are written together thus, *prt*, the meaning is that the values of these letters are to be multiplied together. Thus, *Prt* means that the value of the *P* is to be multiplied by the value of *r*, and that by the value of *t*.

When letters are written in the form of a fraction, thus $\frac{A-P}{Pr}$ the meaning is the same as in common arithmetical fractions; i. e., that the part constituting the numerator is to be divided by the part constituting the denominator.

Thus, $\frac{A-P}{Pr}$ means that the value of *P* is to be subtracted from the value of *A*, and this difference is to be divided by the value of *P* multiplied by the value of *r*.

12. Let $P = \text{Principal}$, $I = \text{Interest}$, $A = \text{Amount}$, $r = \text{rate per unit}$, and $t = \text{time}$ (i. e., number of years).

$$I = Prt \text{ (I.)}$$

$$P = \frac{I}{rt} \text{ (II.)}$$

$$r = \frac{I}{Pt} \text{ (III.)}$$

$$t = \frac{I}{Pr} \text{ (IV.)}$$

$$A = P(1+rt) \text{ (V.)}$$

$$P = \frac{A}{1+rt} \text{ (VI.)}$$

$$r = \frac{A-P}{Pt} \text{ (VII.)}$$

$$= \frac{A-P}{Pr} \text{ (VIII.)}$$

$$t = \frac{n-1}{r} \text{ (IX.)}$$

$$r = \frac{n-1}{t} \text{ (X.)}$$

$$n = tr+1 \text{ (XI.)}$$

Then because $r = \text{interest of } \1 for 1 year , and $t = \text{number of years}$, $rt = \text{interest of } \$1 \text{ for the given time}$, and $Prt = \text{interest of given principal for given time and at given rate}$. Therefore $I = Prt$ and dividing each of these equals, 1st by rt , 2nd by Pt , and 3rd by Pr , we get formulas (II.), (III.) and (IV.) in the margin.

Again, because $rt = \text{interest of } \$1 \text{ at given rate and for given time}$, $1+rt = \text{the amount of } \$1 \text{ at given rate and time}$, and P times $1+rt$, that is, $P(1+rt) = \text{amount of given principal at the given rate and time}$. Therefore $A = P(1+rt)$, which is formula (V.) in the margin, and dividing each of their equals by $1+rt$, we get formula (VI.) in the margin. Taking (V.) and actually multiplying as indicated, the part within the brackets by P , we get $A = P + Prt$; and subtracting P from each of these, we get $A - P = Prt$. Dividing these equals, 1st by Pt and 2nd by Pr , we get formulas (VII.) and (VIII.) in the margin.

Lastly, if we are required to find in what time any sum of money will amount to any given number of times itself at a given rate per cent., or, in other words, in what time any principal will amount to n times that principal where n simply stands for the required number of times, we have in formula (VIII.) in the margin.

$t = \frac{A-P}{Pr} = \frac{nP-P}{Pr}$, because the amount is to be nP ; and dividing both numerator and denominator of this fraction by P , we get formula (IX.) in the margin, multiplying (IX.) by r we get $tr = n-1$; and dividing these equals by t , we get formula (X.); and again, adding 1 to each of these same equals, we get formula (XI.)

APPLICATIONS.

13. When the principal, rate per cent., and time are given, to find the interest—

RULE. $I = Prt$. (i.)

INTERPRETATION.—The interest is found by multiplying the principal by the rate per unit, and the resulting product by the time.

EXAMPLE 1.—What is the interest on \$342.20 for 7 years at 8 per cent.?

OPERATION.

Here $P = \$342.20$, $r = .08$, and $t = 7$.

Then $I = Prt = \$342.20 \times .08 \times 7 = \191.632 . Ans.

14. When the interest, rate per cent., and time are given to find the principal—

RULE. $P = \frac{I}{rt}$ (ii.)

INTERPRETATION.—The principal is found by dividing the interest by the product of the rate per unit and the time.

EXAMPLE 2.—What principal will give \$207.50 interest in $6\frac{1}{2}$ years at $4\frac{1}{2}$ per cent. ?

OPERATION.
Here $I = \$207.50$, $t = 6.5$, and $r = .0475$.

Then $P = \frac{I}{rt} = \frac{\$207.50}{6.5 \times .0475} = \frac{\$207.50}{.30875} = \$672.064$. *Ans.*

15. When the interest, principal, and time are given to find the rate per cent.—

RULE. $r = \frac{I}{Pt}$ (iii.)

INTERPRETATION.—The rate per unit is found by dividing the interest by the product of the principal and time, and the rate per cent. is found from the rate per unit by multiplying the latter by 100.

EXAMPLE 3.—At what rate per cent. will \$729.18 give \$109.11 interest in 9 years ?

OPERATION.
Here $P = \$729.18$, $I = \$109.11$, and $t = 9$.

Then $r = \frac{I}{Pt} = \frac{109.11}{729.18 \times 9} = \frac{109.11}{6562.62} = 0.01662 = \text{rate per unit.}$

Therefore the rate per cent. $= 0.01662 \times 100 = 1.662 = 1\frac{2}{3}$ nearly. *Ans.*

16. When the interest, principal, and rate per cent. are given, to find the time—

RULE. $t = \frac{I}{Pr}$ (iv.)

INTERPRETATION.—The time is found by dividing the interest by the product of the principal and rate per unit.

EXAMPLE 4.—In what time will \$850 give \$89.75 interest, at 13 per cent. ?

Here $P = \$850$, $I = \$89.75$, and $r = .13$.

Then $t = \frac{I}{Pr} = \frac{89.75}{850 \times .13} = \frac{89.75}{110.5} = \frac{8975}{1105} = 0.812217 \text{ years} = 9 \text{ months, } 22 \text{ days.}$

17. When the principal, rate per cent., and time are given, to find the amount—

RULE. $A = P(1+rt)$ (v.)

INTERPRETATION.—The amount is found by multiplying the principal by the amount of \$1 for the given rate and time.

EXAMPLE 5.—To what sum will \$789.80 amount in 11 years, at 3 per cent. ?

OPERATION.
Here $P = \$789.80$, $r = .03$, and $t = 11$.

Then $A = P(1+rt) = \$789.80 \times 1.33 = 1059.434$. *Ans.*

NOTE.— $(1+rt)$ in this question $= 1+.3 \times 11 = 1+.33 = 1.33$.

18. When the amount, rate per cent., and time are given, to find the principal—

$$\text{RULE. } P = \frac{A}{1+rt} \text{ (vi.)}$$

INTERPRETATION.—The principal is found by dividing the given amount by the amount of \$1 for the given time at the given rate.

EXAMPLE 6.—What principal put to interest at $7\frac{1}{2}$ per cent. will amount to \$2000 in 8 years?

$$\begin{array}{l} \text{Here } A = \$2000, r = .075 \text{ and } t = 8. \\ \text{Then } P = \frac{A}{1+rt} = \frac{2000}{1.60} = \frac{20000}{16} = \$1250. \text{ Ans.} \end{array}$$

19. When the amount, principal, and time are given, to find the rate per cent—

$$\text{RULE. } r = \frac{A-P}{Pt} \text{ (vii.)}$$

INTERPRETATION.—The rate per unit is found by subtracting the principal from the amount, and dividing the difference by the principal multiplied by the time. The rate per cent. is found by multiplying the rate per unit by 100.

EXAMPLE 7.—At what rate per cent. will \$730 amount to \$2783.80 in 23 years?

$$\begin{array}{l} \text{Here } A = \$2783.80, P = \$730 \text{ and } t = 23. \\ \text{Then } r = \frac{A-P}{Pt} = \frac{\$2783.80 - \$730}{\$730 \times 23} = \frac{\$2053.80}{\$16790} = .1223 = \text{rate per unit.} \\ \text{Hence rate per cent.} = 12 \cdot 23 = 12\frac{1}{2} \text{ nearly.} \end{array}$$

20. When the amount, principal, and rate per cent. are given, to find the time—

$$\text{RULE. } t = \frac{A-P}{Pr} \text{ (viii.)}$$

INTERPRETATION.—The time is found by subtracting the principal from the amount, and dividing the difference by the principal multiplied by the rate per unit.

EXAMPLE 8.—In what time will \$666.33 amount to \$983.73 at 12 per cent.?

$$\begin{array}{l} \text{Here } A = \$983.73, P = \$666.33 \text{ and } r = .12 \\ \text{Then } t = \frac{A-P}{Pr} = \frac{983.73 - 666.33}{666.33 \times .12} = \frac{317.40}{79.9596} = \frac{317400}{799596} = 3.9695 \text{ years} \\ = 3 \text{ years } 11 \text{ months } 19 \text{ days. Ans.} \end{array}$$

21. *To find the time in which any sum will amount to any given number of times itself at a given rate per cent.—*

$$\text{RULE. } t = \frac{n-1}{r} \text{ (ix.)}$$

INTERPRETATION.—*To find the time in which a given sum will amount to n times itself at a given rate per cent., subtract 1 from n, and divide the remainder by the rate per unit.*

EXAMPLE 9.—In what time will any sum of money amount to eleven times itself at 8 per cent.?

OPERATION.

Here $n=11$ and $r=.08$,

$$\text{Then } t = \frac{n-1}{r} = \frac{11-1}{.08} = \frac{10}{.08} = \frac{1000}{8} = 125 \text{ years. } \textit{Ans.}$$

EXAMPLE 10.—In what time will \$67.83 quadruple itself at $4\frac{2}{3}$ per cent.?

OPERATION.

Here $n=4$, since the money is to quadruple itself, and $r=.0475$.

$$\text{Then } t = \frac{n-1}{r} = \frac{4-1}{.0475} = \frac{3}{.0475} = \frac{30000}{475} = 63.157 \text{ years. } \textit{Ans.}$$

22. *To find the rate per cent. at which any sum will amount to a given number of times itself in a given time—*

$$\text{RULE. } r = \frac{n-1}{t} \text{ (x.)}$$

INTERPRETATION.—*The rate per unit is found by subtracting 1 from n, the number of times itself to which the given principal is to amount, and dividing the remainder by the given number of years.*

EXAMPLE 11.—At what rate per cent. will a given sum amount to 25 times itself in 72 years?

OPERATION.

Here $n=25$ $t=72$.

$$\text{Then } r = \frac{n-1}{t} = \frac{25-1}{72} = \frac{24}{72} = \frac{1}{3} = .33\frac{1}{3} = \text{rate per unit.}$$

Hence rate per cent. = $33\frac{1}{3}$. *Ans.*

23. *To find to how many times itself a given sum will amount in a given time at a given rate per cent.—*

$$\text{RULE. } n = tr + 1. \text{ (xi.)}$$

INTERPRETATION.—*The number of times, or n, is found by multiplying the time by the rate per unit, and adding 1 to the product.*

EXAMPLE 12.—To how many times itself will four cents amount in 20 years at 17 per cent.?

OPERATION.

Here $t = 20$ and $r = 17$.

Then $n = tr + 1 = 20 \times 17 + 1 = 34 + 1 = 44 = 4\frac{4}{5}$ times itself. *Ans.*

EXERCISE.

13. What is the interest on \$723.19 for 7.32 years at 6.7 per cent.
Ans. \$354.6813036.
14. To what sum will \$857.19 amount in $6\frac{1}{2}$ years at $6\frac{1}{2}$ per cent?
Ans. \$1219.352775.
15. To how many times itself will £2 19s. 9½d. amount in 11 years at $72\frac{1}{2}$ per cent.? *Ans.* 8.975, or nearly 9 times.
16. In what time will \$654.32 give \$234.56 interest at 7 per cent.? *Ans.* 5.12112, or 5 years 1 m. 13 days.
17. At what rate per cent. will \$700 amount to \$1200 in 5 years?
Ans. $14\frac{2}{3}$ per cent.
18. In what time will any sum of money quadruple itself at 23 per cent?
Ans. 13 years 15 days.
19. Find the time in which \$270 will give \$87 interest, at 7 per cent.
Ans. 4 years $7\frac{5}{12}$ months.
20. To what sum will \$680 amount in $11\frac{1}{2}$ years, at 11 per cent?
Ans. \$1540.20.
21. What principal will amount to \$2000 in 20 years, at 8 per cent.?
Ans. \$769.23 $\frac{1}{3}$.
22. At what rate per cent. will any sum of money amount to 21 times itself in 24 years?
Ans. $83\frac{1}{3}$ per cent.
23. In what time will a given sum of money amount to 23 times itself, at 16 per cent.?
Ans. $137\frac{1}{2}$ years.
24. Find the interest on \$679.18 at $7\frac{1}{4}$ per cent., for 11.73 years?
Ans. \$617.4255.
25. At what rate per cent. will \$950 amount to \$1763.42 in 10 years?
Ans. 8.562 per cent., or rather over $8\frac{1}{2}$ per cent.
26. In what time will \$666 amount to \$1347.50, at 6 per cent.?
Ans. $17.054\frac{1}{2}$ years, or 17 years 19 days.
27. In what time will \$273 give \$100 interest, at 9 per cent.?
Ans. 4 years 25 days.
28. At what rate per cent. will \$476.30 amount to \$500 in 2 years?
Ans. $21\frac{1}{2}$ per cent.
29. At what rate per cent. will \$749.49 give \$257 interest in 7 years?
Ans. 4.898 per cent.
30. What principal will amount to \$1111.11 in 11 years, at 11 per cent.?
Ans. \$502.7647.
31. Find the interest on £167.47, at 11 per cent. for 9 years.
Ans. £165 15s. $10\frac{11}{12}$ d.

SPECIAL RULES.

24. The interest of \$100 at 6 per cent., for one year, is \$6; hence the interest on \$1 at 6 per cent., for one year, is \$0.06, and for two months it is $\frac{1}{2}$ of \$0.06; i.e., 1 cent.

Hence, to find the interest of \$1 for any number of months, we deduce the following:—

RULE.

Divide the number of months by 2, and call the quotient cents.

EXAMPLE 32.—What is the interest of \$1 at 6 per cent. for 7 years and 9 months?

OPERATION.

7 years and 9 months = 93 months, and $93 \div 2 = 46\frac{1}{2}$ cents = \$0.465. *Ans.*

EXAMPLE 33.—Find the interest on \$72.93 for 7 years and 8 months at 6 per cent.

OPERATION.

7 years 8 mo. = 92 months, half of 92 = 46 cents = interest of \$1 for given rate and time.

Then $\$0.46 \times 72.93 = \33.5478 . *Ans.*

EXERCISE.

34. Find the interest on \$1 for 11 months at 6 per cent.
Ans. $5\frac{1}{2}$ cents.
35. Find the interest on \$1 for 16 months at 6 per cent.
Ans. \$0.08, or 8 cents.
36. Find the interest on \$1 for 9 years 8 months at 6 per cent.
Ans. \$0.58.
37. What is the interest on \$1 for 16 yrs. 3 months at 6 per cent.?
Ans. \$0.97 $\frac{1}{2}$.
38. What is the interest on \$1 for 11 yrs. 7 months at 6 per cent.?
Ans. \$0.695.
39. What is the interest on \$1 for 12 yrs. 5 months at 6 per cent.?
Ans. \$0.745.
40. Find the interest on \$279.40 for 3 yrs. 2 mo's. at 6 per cent.
Ans. \$53.086.
41. Find the interest on \$189.70 for 6 yrs. 7 mo's. at 6 per cent.
Ans. \$74.9315.
42. Find the interest on \$1463 for 3 yrs. 11 mo's. at 6 per cent.
Ans. \$343.805.
43. Find the interest on \$28967.50 for 11 years 1 month at 6 per cent.
Ans. \$19263.3875.

25. Since in computing interest the month is taken as 30 days, two months will contain 60 days, and, by Art. 24, the interest on \$1 at 6 per cent. for 2 months or 60 days is one cent, the interest on \$1 at 6 per cent. per annum, for 6 days, will therefore be $\frac{1}{10}$ of one cent; i. e., one mill or $\frac{1}{1000}$ of \$1.

Hence, to find the interest on \$1 at 6 per cent. per annum for days, we have the following:—

RULE.*

Call one-sixth of the number of days mills or thousandths of a dollar.

EXAMPLE 41.—What is the interest on \$1 at 6 per cent. for 16 days?

OPERATION.

$$16 \div 6 = 2\frac{2}{3} \text{ mills} = \$0.0026. \text{ Ans.}$$

EXERCISE.

45. What is the interest on \$1 for 2 days at 6 per cent. ?
Ans. \$0.0003.
46. What is the interest on \$1 for 7 days at 6 per cent. ?
Ans. \$0.001 $\frac{1}{2}$.
47. What is the interest on \$1 for 11 days at 6 per cent. ?
Ans. \$0.001 $\frac{1}{3}$.
48. What is the interest on \$1 for 27 days at 6 per cent. ?
Ans. \$0.004 $\frac{1}{2}$.
49. What is the interest on \$1 for 47 days at 6 per cent. ?
Ans. \$0.007 $\frac{2}{3}$.
50. Required the interest on \$1 for 8 months 12 days at 6 per cent.
Ans. \$0.042.
51. Required the interest on \$1 for 66 days at 6 per cent.
Ans. \$0.011.
52. Required the interest on \$1 for 2 years 2 months 19 days at 6 per cent.
Ans. \$0.133 $\frac{1}{2}$.
53. Find the interest on \$1 for 7 years 8 months 9 days at 6 per cent.
Ans. \$0.461 $\frac{1}{2}$.
54. What is the interest on \$1 for 17 years 11 months 23 days at 6 per cent. ?
Ans. \$1.078 $\frac{2}{3}$.
55. Required the interest on \$1 for 12 years 7 months 17 days at 6 per cent.
Ans. \$0.757 $\frac{2}{3}$.

26. To find the interest on any sum of money at 6 per cent. per annum for any time:—

RULE.

Find the interest on \$1 for the given time, by Arts. 24 and 25, and multiply this by the given principal.

EXAMPLE 56.—What is the interest on \$763.20 at 6 per cent. for 6 years 7 months and 26 days ?

* This is the method in common use for computing interest for days: but, since it considers the year as containing only 360 days instead of 365, the result is too large by $\frac{5}{365}$, or $\frac{1}{73}$ of itself. Hence, when perfect accuracy is desired, the interest for the days when obtained by the rule must be diminished by $\frac{1}{73}$ part of itself.

OPERATION.

| | |
|--------------------------------------|-------------------|
| Interest on \$1 for 6 years 7 months | = \$0.395 |
| Interest on \$1 for 26 days | = $\frac{4}{100}$ |

Therefore interest on \$1 for 6 yrs. 7 months 26 days = \$0.399 $\frac{1}{2}$

Then* \$0.399 $\frac{1}{2}$ \times 763.20 = \$304.7712. *Ans.*

EXERCISE.

57. Find the interest on \$917.30 for 7 months 17 days at 6 per cent. *Ans.* \$34.704516.
58. Find the interest on \$842.50 for 3 months 13 days at 6 per cent. *Ans.* \$14.462916.
59. Required the interest on \$573.83 at 6 per cent. for 2 years 11 months 10 days. *Ans.* \$101.3766.
60. Required the interest on \$642.30 at 6 per cent. for 6 years 9 months 19 days. *Ans.* \$262.16545.
61. Required the interest on \$1427.87 $\frac{1}{2}$ at 6 per cent. for 5 years 5 months 7 days. *Ans.* \$465.7252.
62. Find the interest on \$709.63 for 4 years 7 months 16 days at 6 per cent. *Ans.* \$197.040596.
63. Find the amount of \$2463.20 at 6 per cent. for 7 years 7 months 22 days. *Ans.* \$3592.9877.
64. What is the interest on \$999.99 at 6 per cent. for 9 years 9 months 9 days? *Ans.* \$586.494135.
65. What is the interest on \$68.70 for 3 years 4 months 27 days at 6 per cent.? *Ans.* \$14.04915.
66. Find the interest on \$742.63 at 6 per cent. for 3 years 28 days. *Ans.* \$137.139.
67. To what sum will \$200 amount in 7 years 4 months 11 days at 6 per cent.? *Ans.* \$288.366.
68. To what sum will \$743.63 amount in 9 years 3 months 9 days at 6 per cent.? *Ans.* \$1157.460095.

27. To find the interest on any sum at any other rate per cent. for any given time—

RULE.

Find the interest on the given principal for the given time at 6 per cent. by Art. 26.

Then add to or subtract from this interest such a fractional part of itself as the given rate exceeds or falls short of 6 per cent. per annum.

The amount is obtained by adding the interest and the principal together.

* In order to obtain the correct answer, this fraction when it occurs must be retained in the form of a vulgar fraction; and in that case it is better to make the interest of \$1 for the given time the multiplier.

EXAMPLE 69.—What is the interest on \$450 for 3 years 6 months 11 days at 8 per cent ?

OPERATION.

Interest on \$1 at 6 per cent. for given time = $\$0.211\frac{5}{8}$.

Interest on \$450 at 6 per cent. for given time = $\$0.211\frac{5}{8} \times 450 = \95.325 .

Hence interest on \$450 at 8 per cent. for given time = $\$95.325 + \text{one third of } \$95.325 = \$127.10$. *Ans.*

NOTE.—Since $8 = 6 + 2 = 6 + \frac{1}{3}$ of 6 we find the interest at 6 per cent., and increase it by *one third* of itself for the interest at 8 per cent.

So for interest at 9 per cent., we should find the interest at 6 per cent., and increase it by *one-half* of itself; for 7 per cent., increase the interest at 6 per cent by *one-sixth*; at 14 per cent., double the interest at 6 per cent., and increase it by $\frac{1}{2}$ of the interest at 6 per cent.; at 5 per cent., find the interest at 6 per cent. and deduct *one-sixth*; at $4\frac{1}{2}$ per cent., find the interest at 6 per cent. and deduct *one-fourth*, &c., &c.

EXERCISE.

70. Required the interest on \$1234.56 for 8 years 9 months 10 days at 7 per cent. *Ans.* \$758.5685.
71. Required the interest on \$9876.54 for 2 years 1 month 11 days at 3 per cent. *Ans.* \$626.337245.
72. Required the interest on \$715.30 for 3 years 7 months 10 days at 8 per cent. *Ans.* \$206.6422.
73. To what sum will \$555.55 amount in 2 years 4 months 8 days at 12 per cent. ? *Ans.* \$712.58546.
74. To what sum will \$7766.55 amount in 100 days at 5 per cent. ? *Ans.* \$7874.41875.
75. To what sum will \$500 amount in 8 years 8 months 8 days at 16 per cent. ? *Ans.* \$1195.111.
76. What is the interest on \$576 for 3 years 5 months 7 days at 5 per cent. ? *Ans.* \$98.96.
77. What is the interest on \$2478.91 for 2 years 6 months 11 days at $4\frac{1}{2}$ per cent. ? *Ans.* \$282.285.
78. What is the interest on \$780 from May 9, to December 11, at 6 per cent. ? *Ans.* \$28.08.
79. What is the interest on a note of \$1830.63 from August 16, 1851, to June 19, 1852, at 7 per cent. ? *Ans.* \$109.63439.
80. What is the amount of a note of \$6200 from Sept. 3, 1858, to January 9, 1859, at 6 per cent. ? *Ans.* \$6332.266.

PARTIAL PAYMENTS.

28. To compute the interest on notes or bonds, when partial payments have been made:—

RULE.

*If the interest be paid by days :
Multiply the sum by the number of days which have elapsed before any payment was made. Subtract the first payment, and multiply*

the remainder by the number of days which passed between the first and second payments. Subtract the second payment, and multiply this remainder by the number of days which passed between the second and third payments. Subtract the third payment, &c.

Add all the products together, and find the interest of their sum for one day.

If the interest is to be paid by the week or month, substitute weeks or months for days, in the above rule.

EXAMPLE 81.—How much principal and interest have I to pay on the following note on the 10th November, 1859?

TORONTO, 18th October, 1858.

For value received, I promise to pay to Timothy Thomas, or order, the sum of six hundred and twenty dollars, on demand, with interest at 6 per cent.

THOMAS WILLIAMS.

The following endorsements were made on this note :

| | | | |
|----------------------|--------------------|--------|-------|
| 1858.—November 25th, | there was endorsed | \$ | 47-50 |
| “ December 28th, | “ “ “ | 108-93 | |
| 1859.—February 11th, | “ “ “ | 216-18 | |
| “ June 6th, | “ “ “ | 60-10 | |
| “ September 2nd, | “ “ “ | 183-25 | |

OPERATION.

| | | | |
|------------------------------------|-----------|-----|-------|
| From 18th October to 25th November | there are | 38 | days. |
| “ 25th Nov. to 28th December | “ | 33 | “ |
| “ 28th Dec. to 11th February | “ | 45 | “ |
| “ 11th February to 6th June | “ | 115 | “ |
| “ 6th June to 2nd September | “ | 88 | “ |
| “ 2nd September to 10th Nov. | “ | 69 | “ |

Whole sum \$620 for 38 days = \$23560 for 1 day.

First endorsement 47-50

Balance \$572-50 for 33 days = \$18892-50 for 1 day.

Second endorsement 108-93

Balance \$463-57 for 45 days = \$20860-65 for 1 day.

Third endorsement 216-18

Balance \$247-39 for 115 days = \$28349-85 for 1 day.

Fourth endorsement 60-10

Balance \$187-29 for 88 days = \$16481-52 for 1 day.

Fifth endorsement 183-25

Balance \$4-04 for 69 days = 278-76 for 1 day.

Whole interest = that of \$108523-28 for 1 day.

Interest on \$108523-28 at 6 per cent, for 1 year = \$6511-3968

Hence interest for 1 day = \$6511-3968 ÷ 365 = \$17-8394

Then interest due = \$17-8394

Balance on note = 4-04

Principal and interest due = \$218-794

EXERCISE.

82. What principal and interest was due on the following note on the 7th October, 1860 ?

GUELPH, *June 2nd*, 1859.

For value received, I promise to pay, on demand, to James George, or order, the sum of twelve hundred and seventeen dollars and thirty cents, with interest from date at 6 per cent.

JOSEPH JOHNS.

On this note there were endorsed the following payments :

| | | |
|-------------------|----------|-----------|
| 1859.—July 17th, | received | \$207-80. |
| “ Oct. 6th, | “ | 209-60 |
| “ Dec. 11th, | “ | 320-90 |
| 1860.—March 29th, | “ | 421-83 |

Ans. \$98-6816.

83. What principal and interest was due on the following note on the 1st May, 1863 ?

PORT HOPE, *June 17th*, 1860.

For value received, I promise to pay, on demand, to Messrs. Henly & Jobson, or order, the sum of seven thousand, three hundred and forty-eight dollars and twenty-five cents, with interest from date at 8 per cent.

HENRY GOODPAY.

On this note there were endorsed the following payments :

| | | |
|----------------------|----------|-----------|
| 1860.—September 5th, | received | \$2463-80 |
| “ December 7th, | “ | 392-20 |
| 1861.—June 11th, | “ | 982-20 |
| 1862.—February 7th, | “ | 2842-90 |
| “ December 19th, | “ | 317-23 |

Ans. \$1003-1333.

 COMPOUND INTEREST.

29. In the present article we shall merely take some of the simpler problems in Compound Interest, leaving the full discussion of the rule until after the pupil is familiar with the use of Logarithms. (See Sect. XI.)

30. We have seen (Art. 10) that when money is lent at compound interest, the interest is added to the principal at the close of each period, and, with it, constitutes a new principal for the next term.

Hence to find the compound interest of any sum for any given time at a given rate per cent :—

RULE.

Find the interest on the given principal for one period, *i. e.*, ONE YEAR, HALF YEAR, or QUARTER, as the case may be, and add it to the principal.

Then find the interest on this amount for the NEXT PERIOD and add it to the principal used for that period, as before.

Proceed in this manner with each successive year or period of the proposed time.

Then the last result will be the amount of the given principal, at the given rate, for the given time. Subtract the given principal from this, and the remainder will be the Compound Interest required.

EXAMPLE 1.—What is the Compound Interest on \$1000 for 4 years at 5 per cent. per annum?

OPERATION

| | | |
|--------------|--|--|
| \$1000 | Principal. | |
| 50 | Interest for 1st year. | |
| \$1050 | Amount for 1 year = principal for 2nd year. | |
| 32 50 | Interest for 2nd year. | |
| \$1102 50 | Amount for 2 years = principal for 3rd year. | |
| 55 125 | Interest for 3rd year. | |
| \$1157 625 | Amount for 3 years = principal for 4th year | |
| 57 84125 | Interest for 4th year. | |
| \$1215 50625 | Amount for 4 years. | |
| 1000 | given Principal. | |

Ans. \$215 50625 = Compound Interest required.

EXERCISE.

2. What is the Compound Interest of \$1800 for 5 years at 6 per cent. per annum? Ans. \$608 806.
3. What is the Compound Interest of \$700 for 3½ years at 7 per cent. half-yearly? Ans. \$424 040.

NOTE.—Since the payments are made *half-yearly*, and bear interest at the rate of 7 per cent. per half year, we simply find the amount of the given principal at 7 per cent. for 7 payments.

4. What are the amount and Compound Interest of \$673 40 for 2 years at 3 per cent. quarterly?
Ans. \$853 0429 = Amount. \$179 6429 = Interest.
5. What are the amount and Compound Interest of \$860 for 3 years at 4 per cent. half-yearly?
Ans. \$1088 1743 = Amount. \$228 1743 = Interest.

31. Compound Interest is most expeditiously calculated by the following—

TABLE

SHEWING THE AMOUNTS OF \$1 or £1 AT COMPOUND INTEREST, FOR ANY NUMBER OF PAYMENTS FROM 1 TO 60.

| No. of Pay- ments. | 3 per cent. | 4 per cent. | 5 per cent. | 6 per cent. | No. of Pay- ments. | 3 per cent. | 4 per cent. | 5 per cent. | C | £ |
|-----------------------|-------------------|-------------------|-------------------|-------------------|-----------------------|-------------------|-------------------|-------------------|----|-------|
| 1 | 1'03000 | 1'04000 | 1'05000 | 1'06000 | 26 | 2'15659 | 2'77247 | 3'55567 | 4 | 5998 |
| 2 | 1'06090 | 1'08160 | 1'10250 | 1'12360 | 27 | 2'22129 | 2'88337 | 3'73346 | 4 | 8235 |
| 3 | 1'09275 | 1'12486 | 1'15762 | 1'19109 | 28 | 2'28793 | 2'99870 | 3'92013 | 5 | 11169 |
| 4 | 1'12551 | 1'16986 | 1'21551 | 1'26245 | 29 | 2'35657 | 3'11865 | 4'11614 | 5 | 41839 |
| 5 | 1'15927 | 1'21665 | 1'27628 | 1'33825 | 30 | 2'42726 | 3'24840 | 4'32194 | 5 | 74848 |
| 6 | 1'19405 | 1'26532 | 1'34010 | 1'41855 | 31 | 2'50008 | 3'37313 | 4'53804 | 6 | 96810 |
| 7 | 1'12987 | 1'31593 | 1'40710 | 1'50362 | 32 | 2'57508 | 3'50806 | 4'76494 | 6 | 45339 |
| 8 | 1'26677 | 1'36857 | 1'47745 | 1'59385 | 33 | 2'65233 | 3'64838 | 5'00319 | 6 | 84059 |
| 9 | 1'30477 | 1'42331 | 1'55133 | 1'68948 | 34 | 2'73190 | 3'79432 | 5'25335 | 7 | 25102 |
| 10 | 1'34392 | 1'48024 | 1'62889 | 1'79085 | 35 | 2'81386 | 3'94609 | 5'51601 | 7 | 68609 |
| 11 | 1'38422 | 1'53945 | 1'71034 | 1'89836 | 36 | 2'89828 | 4'10393 | 5'79182 | 8 | 14725 |
| 12 | 1'42576 | 1'60103 | 1'79586 | 2'01226 | 37 | 2'98523 | 4'26809 | 6'08141 | 8 | 63609 |
| 13 | 1'46853 | 1'66507 | 1'88565 | 2'13293 | 38 | 3'07478 | 4'43881 | 6'38548 | 9 | 15425 |
| 14 | 1'51259 | 1'73168 | 1'97993 | 2'26090 | 39 | 3'16703 | 4'61637 | 6'70475 | 9 | 70351 |
| 15 | 1'55797 | 1'80094 | 2'07893 | 2'39656 | 40 | 3'26204 | 4'80102 | 7'03999 | 10 | 28572 |
| 16 | 1'60471 | 1'87298 | 2'18287 | 2'54035 | 41 | 3'35990 | 4'99306 | 7'39169 | 10 | 90286 |
| 17 | 1'65285 | 1'94790 | 2'29202 | 2'69277 | 42 | 3'46070 | 5'19278 | 7'76159 | 11 | 55703 |
| 18 | 1'70243 | 2'02582 | 2'40662 | 2'85434 | 43 | 3'56452 | 5'40049 | 8'14967 | 12 | 25045 |
| 19 | 1'75351 | 2'10685 | 2'52695 | 3'02560 | 44 | 3'67145 | 5'61651 | 8'55715 | 12 | 93548 |
| 20 | 1'80611 | 2'19112 | 2'65333 | 3'20713 | 45 | 3'78160 | 5'84118 | 8'98501 | 13 | 76461 |
| 21 | 1'86029 | 2'27877 | 2'78596 | 3'39956 | 46 | 3'89504 | 6'07482 | 9'43426 | 14 | 59049 |
| 22 | 1'91610 | 2'36992 | 2'92526 | 3'60354 | 47 | 4'01190 | 6'31782 | 9'90597 | 15 | 46592 |
| 23 | 1'97359 | 2'46472 | 3'07152 | 3'81975 | 48 | 4'13225 | 6'57053 | 10'40127 | 16 | 39387 |
| 24 | 2'03279 | 2'56330 | 3'22510 | 4'04893 | 49 | 4'25622 | 6'83335 | 10'92133 | 17 | 37700 |
| 25 | 2'09378 | 2'66584 | 3'38635 | 4'29187 | 50 | 4'38391 | 7'10668 | 11'46740 | 18 | 42515 |

32. To Compute Compound Interest by the above Table:—

RULE.

Find by the table the amount of \$1 for the given time and at the given rate.

Multiply the sum thus found by the given principal, and the result will be the required amount.

Subtract the principal from this amount, and the remainder will be the Compound Interest.

EXAMPLE 6.—What are the amount and compound interest of \$3400 at 5 per cent. for 15 years?

OPERATION.

By the table the amount of \$1 at 5 per cent. for 15 years = \$2'07893.

Then \$2'07893 × 3400 = \$7068'362 = Amount.

3400 Principal.

\$3668'362 = Interest

EXAMPLE 7.—What is the amount and compound interest of £47 10s. for 6 years, at 3 per cent. half-yearly?

OPERATION,
£47 10s. = £47.5.

We find by the table that

£142576 is the amount of £1 for the given time and rate.

47.5 is the multiplier.

£67 7 236 = 67 14 5½ is the required amount.

47 10 0 is the given principal.

And £20 4 5½ is the required interest.

EXERCISE.

8. What are the amount and compound interest on \$875 for 11 years at 6 per cent. ?
Ans. Amount = \$1661.0125.
Interest = \$786.0125.
9. What are the amount and compound interest on \$643.98 for 13 years at 4 per cent. half-yearly ?
Ans. Amount = \$1785.41523.
Interest = \$1141.43523.
10. What are the amount and compound interest of 1 cent at 6 per cent. per annum for 45 years ? *Ans.* Amount = \$137646.
Interest = \$127646.
11. What are the amount and compound interest of \$78.20 for 7 years at 3 per cent. quarterly ? *Ans.* Amount = \$178.916.
Interest = \$100.716.
12. What are the amount and compound interest of \$777.77 for 9 years at 5 per cent. half-yearly ?
Ans. Amount = \$1871.7968.
Interest = \$1094.0268.
13. What are the amount and compound interest of £44 5s. 9d. for 11 years at 6 per cent. per annum ?
Ans. Amount = £84 1s. 5d.
Interest = £39 15s. 8d.
14. What are the amount and compound interest of £32 4s. 9½d. for 3 years at 4 per cent. half-yearly ?
Ans. Amount = £40 15s. 10½d. nearly.
Interest = £ 8 11s. 1 d.

33. Given the amount, time, and rate—to find the principal; that is, to find the *present worth* of any sum to be due hereafter—a certain rate of interest being allowed for the money now paid—

RULE.

Find by the Table the amount of \$1 at the given rate and for the given time, and divide it into the given amount. The quotient will be the principal.

EXAMPLE 15.—What principal will amount to \$10000 in 12 years at 6 per cent. compound interest?

OPERATION.

Amount of \$1 for 12 years at six per cent. = \$2.0122.
 \$10000 ÷ 2.0122 = \$4969.684. *Ans.*

EXERCISE.

16. What principal will amount to \$7439.87 in 7 years at 4 per cent. compound interest? *Ans.* \$5653.697.
17. What principal will amount to \$9193.90 in 20 years at 5 per cent. compound interest? *Ans.* \$3465.081.
18. What ready money ought to be paid for a debt of £595 10s. 2½d. to be due 3 years hence, allowing 6 per cent. per annum compound interest? *Ans.* £500.
19. What ready money ought to be paid for a debt of \$7111.11, to be due 7 years hence, allowing 6 per cent. compound interest? *Ans.* \$4729.295.
20. What principal, put to interest for 6 years, would amount to £268 0s. 4½d., at 5 per cent. per annum? *Ans.* £200.

DISCOUNT.

34. Discount is an allowance made for the payment of a debt before it is due.

35. The present worth of a debt, payable at some future time, without interest, is that sum of money which, being put out at legal interest, will amount to the debt by the time it becomes due.

Thus, if I owe a man \$100 and give him a note for that amount, payable one year hence, without interest, the *present* value of my note is less than \$100, since \$100 being put out at interest for 1 year at 6 per cent. will amount to \$106.

36. From Art. 18 it is evident that to find the present worth of a note, payable at some future time, without interest, is simply to find what principal, put to interest at the rate specified, will amount to the sum named on the face of the note in the given time; i. e., by the time the note becomes due.

Hence, to find the present worth of any sum, to be paid at some future time, without interest, we have (Art. 18) the following:—

$$\text{RULE. } P = \frac{A}{1+rt}$$

INTERPRETATION.—The present worth is found by dividing the amount of the note, debt, &c., by the amount of \$1, at the specified rate per cent. for the given time.

NOTE.—The discount is found by deducting the present value from the note, debt, &c.

EXAMPLE 1.—What is the present value of a note for \$860 payable 3 years hence, allowing discount at the rate of 6 per cent. per annum?

OPERATION.

Here $A = \$860$, $r = .06$, and $t = 3$. Whence $1+rt = 1.18$.

$$\text{Then } P = \frac{A}{1+rt} = \frac{860}{1.18} = \$728^s 81^{\frac{1}{2}} \frac{1}{2} \frac{1}{2}. \text{ Ans.}$$

PROOF.—Interest on $\$728^s 81^{\frac{1}{2}} \frac{1}{2} \frac{1}{2}$ for 3 years at 6 per cent. = $\$131^s 18^{\frac{3}{4}}$.

Added principal..... = $728^s 81^{\frac{1}{2}} \frac{1}{2} \frac{1}{2}$.

Amount..... = $\$860^00$

EXAMPLE 2.—What is the discount on a note for \$728.63 due 9 months hence, allowing discount at 7 per cent. per annum?

OPERATION.

Here $A = \$728.63$, $r = .07$, and $t = .75$ year. Whence $1+rt = 1.0525$.

$$\text{Then } P = \frac{A}{1+rt} = \frac{728.63}{1.0525} = \$692.2858 \text{ present worth.}$$

Then amount on face of note..... \$728.63

Present value..... 692.285

Discount.....\$ 36.344. Ans.

EXERCISE.

- What is the present worth of a note for \$962, payable in one year, at 4 per cent. discount? *Ans.* \$925.
- What is the present worth of \$2202, payable in 5 years and 9 months, at 6 per cent. per annum discount? *Ans.* \$1637.174.
- What sum will discharge a debt of \$1093.50, to be due in 8 months hence, allowing 6 per cent. per annum discount? *Ans.* \$964.9038.
- What ready money will now pay a debt of \$716 due 7 months hence, allowing discount at 8 per cent.? *Ans.* \$684.0764.
- What ready money will now pay a debt of \$1342.50, due 125 days hence, at $6\frac{1}{2}$ per cent.? *Ans.* \$1313.266.
- If a legacy of \$2400 is left to me on the 3rd of May, to be paid on the Christmas day following, what must I receive as present payment, allowing 5 per cent. per annum discount? *Ans.* \$2324.84.
- Find the discount on a bill of \$2202 at 5 per cent., payable 9 months hence. *Ans.* \$79.59036.
- What is the present worth of a note for \$4360, payable one year and 5 months hence, at 6 per cent.? *Ans.* \$4018.43317.
- What is the present worth of a note for \$1647, due 11 months hence, at 6 per cent.? *Ans.* \$1561.13744.

12. Required the present worth of a note for \$2000, due 3 years 7 months hence, at 6 per cent. *Ans.* \$1646·09053.
13. What is the discount on a note for \$2070·90, payable 1 year 7 months hence, at 5 per cent. ? *Ans.* \$151·919.
14. What is the present worth of a note of \$970·63, payable in 11 months, at 8 per cent. ? *Ans.* \$904·313.
- NOTE.—When the payments are to be made at different times, find the present value of the sums separately; their sum will be the present value of the note, and, as before, this subtracted from the whole amount will give the discount.
15. What is the discount on \$3024, the one-half payable in 6, and the remainder in 12 months, 7 per cent. per annum being allowed ? *Ans.* \$150·0464.
16. A merchant owes \$440, payable in 20 months, and \$896, payable in 24 months; the first he pays in 5 months, and the second in one month after that. What did he pay, allowing 8 per cent. per annum ? *Ans.* \$1200.

BANK DISCOUNT.

37. Bank Discount is a charge made by a bank for the payment of money on a note before the note is due, and differs materially from discount as commonly calculated.

38. Banks consider the discount to be the same as the interest on the whole amount of the note, from the time it is discounted until the time it becomes due. Bank Discount is therefore greater than the true discount by the interest on the discount.

39. The three days of grace which, by mercantile usage, are allowed to elapse after a note falls due, before it is payable, are always included by banks in the time for which they calculate the discount.

40. Two kinds of notes are discounted at banks:

1st. *Business notes, or business paper.* These are notes actually given by one individual to another for property sold or value received.

2nd. *Accommodation notes, called also accommodation paper.* These are notes made for the purpose of borrowing money from the banks.

41. To find the bank discount on a note:—

RULE.

Add 3 days to the time which the note has to run before it becomes due, and calculate the interest for this time at the given rate per cent.

EXAMPLE 17.—What is the bank discount on a note of \$700, payable in 70 days, allowing discount at 6 per cent. ?

OPERATION.

Here the time the note has to run is 73 days = 2 months 12 days.

Interest of \$1 at 6 per cent. for 2 months 12 days, is \$0.012.

Interest of \$700 at 6 per cent. for 2 months 12 days = \$0.012 \times 700 = \$8.40. *Ans.*

EXERCISE.*

18. What is the bank discount on a note for \$986, having 2 years and 3 months to run, allowing discount at 7 per cent.?
Ans. \$155.8701.
19. If I have a note for \$640, payable in 100 days, and get it discounted at the rate of 8 per cent. per annum, what discount am I charged?
Ans. \$14.6488.
20. I sell a horse and carriage for \$563.80, and receive a note for that sum, payable, without interest, 91 days hence. Now if I get this discounted at the rate of 6 per cent. per annum, what sum do I receive?
Ans. \$554.967.

42. It is often necessary to make a note of which the present value shall be a certain sum.

Thus, suppose I require to receive from the bank \$1000, and wish to give my note, payable in 7 months, at 6 per cent., what amount must I put on the face of the note?

Now the interest on \$1 at 6 per cent. for 7 months and 3 days (i. e. days of grace) is \$0.0355, and this will be the bank discount on \$1 for 7 months at 6 per cent.

To get the present value of \$1, we subtract \$0.0355 from \$1, which gives us \$0.9645.

Hence, for every \$0.9645 I receive, I must put \$1 on the face of the note; and therefore to receive \$1000, I must put $\frac{1000}{0.9645}$, i. e. \$1036.806 on the face of the note.

PROOF.—Face of note..... \$1036.806
Bank discount on \$1036.806 at 6 per cent. per an. for 7 m. 36.806
Present value..... \$1000.00

Hence to find the face of a note, due at some future time and discounted at a given rate per cent. per annum, that shall have a known present value, we have the following:—

*These examples are worked by the rule given in Arts. 26 and 27. If the absolutely correct answer is required, it must be obtained by deducting from these results $\frac{1}{3}$ of the interest for the days used, as before explained. In example 19, this will be observed, it makes a difference of 20 cents.

RULE.

Find the present value of \$1 for the same time (adding the three days of grace) and at the same rate; divide the required present value of the note by this, and the quotient will be the face of the note.

EXAMPLE 21.—For what sum must a note be drawn at 8 months 18 days, so that discounted immediately at 6 per cent. it shall produce \$670?

OPERATION.

Interest on \$1 for 8 months 21 days at 6 per cent. = \$0.0435, and this taken from \$1 gives us \$0.9565 = present worth of \$1.

Then $\frac{670}{0.9565} = \$700.47$. *Ans.*

EXERCISE.*

22. What sum must I put on the face of a note payable in 90 days so that I may obtain \$3755 when discounted at a bank at 7 per cent. ? *Ans.* \$3824.15.
23. For what sum must a note be drawn payable in 6 months in order that its proceeds at 5 per cent. bank discount may be \$1147.80 ? *Ans.* \$1177.734.
24. For what sum must a note be drawn payable in 45 days so that its proceeds at $3\frac{1}{2}$ per cent. bank discount may be \$713.90 ? *Ans.* \$717.2471.

EQUATION OF PAYMENTS. 20

43. Equation of Payments is the process of finding the *equated* or *average time* when two or more payments, due at different times, may be made *at once* without loss to either party.

44. The average time for the payment of several sums due at different times is called the *mean time* or *equated time*.

45. To find the equated time for any number of payments:—

RULE.†

First multiply each debt by the time before it becomes due; then divide the sum of the products thus obtained by the sum of the payments, and the quotient will be the equated time required.

* Work by Arts. 26 and 27.

† This rule is based upon the supposition that what is gained by keeping certain payments after they become due is equal to what is lost by paying other payments before they become due. This, however, is not exactly true: for the gain is the interest, while the loss is equal only to the

NOTE.—When there are both days and months, they must all be reduced to the same unit; *i. e.*, the payments must all be reckoned for so many days, or so many months or parts of a month. If one of the payments is due on the day from which the equated time is reckoned, the corresponding product will be nothing; but in finding the *sum* of the debts, this payment must be added with the others. (See Example 3 below.)

EXAMPLE 1.—A merchant purchases a vessel for \$7000, \$2000 to be paid in 3 months, \$2000 in 5 months, and the balance in 11 months. Now if he wishes to make the whole in one payment for what time must his note be drawn?

OPERATION.

$$\begin{array}{r} \$2000 \times 3 = \$ 6000 \times 1 \\ 2000 \times 5 = 10000 \times 1 \\ 3000 \times 11 = 33000 \times 1 \\ \hline 7000) \quad \$49000(7 \text{ months. } \textit{Ans.} \end{array}$$

EXPLANATION.—The interest of \$2000 for 3 months is equal to the interest of \$6000 for one month. Similarly, the interest of the second payment is equal to the interest of \$10000 for one month, and the interest of the third payment is equal to the interest of \$33000 for one month. Hence, the interest of the several payments, at the given times, will be equal to that of \$49000 for one month; and if we divide this \$49000 by the sum of the payments, \$7000, we obtain 7 months for the equated time.

$$\text{That is, } \$7000 : \$49000 :: 1 \text{ month : } \textit{Ans.} = \frac{\$49000 \times 1}{\$7000} = 7 \text{ months.}$$

EXAMPLE 2.—A person owes another £20, payable in 6 months; £50, payable in 8 months; and £90, payable in 12 months. At what time may all be paid together, without loss or gain to either party?

OPERATION.

$$\begin{array}{r} \text{£} \qquad \qquad \text{£} \\ 20 \times 6 = 120 \\ 50 \times 8 = 400 \\ 90 \times 12 = 1080 \\ \hline 160 \quad 160)1600(10 \text{ months. } \textit{Ans.} \\ \quad \quad \quad 160 \end{array}$$

discount, which (Art. 38) is always less than the interest: but the discrepancy is so trifling as not to make any material difference in the result.

With this exception, the rule is true, and may be demonstrated as follows:—Let p = first payment, and t = the time before it becomes due; p' = other payment, and t' = the time before it becomes due; x = equated time, and r = rate of interest per unit.

And since x , the equated time, lies between t and t' , the time between t and x is $x - t$, and that between t' and x is $t' - x$.

The interest of p for the time $x - t$ is (from Art. 13) $pr(x - t)$.

Also interest of p' for the time $t' - x$ is $p'r(t' - x)$.

Hence $pr(x - t) = p'r(t' - x)$.

And $x = \frac{p't + p't'}{p + p'}$, which is the rule, and may be similarly proved for any number of payments.

SIMPLE PARTNERSHIP.

50. When the partners employ their shares of the capital for the same period of time, the partnership is called Simple Partnership.

It is also called Single Partnership, or Partnership without Time.

51. It is evident that the whole stock which suffers the gain or loss must bear the same proportion to the stock of each partner that the whole gain or loss bears to his share of the gain or loss.

Hence, for partnership without time, we have the following:—

RULE.

As the whole stock is to each man's share of the stock, so is the whole gain or loss to each man's share of the gain or loss.

EXAMPLE 1.—A and B enter into trade with a capital of \$3700, of which A contributes \$2000 and B the remainder. They gain \$1200. What is each man's share of the profits?

OPERATION.

Whole stock : A's stock :: whole profit : A's profit.

That is, \$3700 : \$2000 :: \$1200 : $\frac{2000 \times 1200}{3700} = \$648645 = \text{A's share.}$

Again, whole stock : B's stock :: whole profit : B's profit.

That is, \$3700 : \$1700 :: \$1200 : $\frac{1700 \times 1200}{3700} = \$551351 = \text{B's share.}$

NOTE.—After A's share has been found, B's share may be obtained by subtracting A's profit from the whole profit.

EXERCISE.

- Two merchants enter into partnership with a stock of \$4300, of which A contributes \$3000. They gain \$1117; how should this be divided between them?
Ans. A's share = \$779-302.
 B's share = \$337-697.
- Three persons, A, B and C, agree to form a company for the manufacture of woollen cloths. A puts in \$6470, B \$3780, and C \$9860. By the end of the year they find that they have gained \$7890. What portion of this profit belongs to each?
Ans. A's share = \$2538-453,
 B's share = \$1483-053,
 C's share = \$3868-493.
- B and C buy certain merchandize, amounting to \$320, of which B pays \$120, and C \$200; and they gain \$80. How is it to be divided?
Ans. B \$30 and C \$50.
- B and C gain by trade \$728; B put in \$1200, and C \$1600. What is the gain of each?
Ans. B \$312 and C \$416.
- Two persons are to share \$100 in the proportions of 2 to B and 1 to C. What is the share of each?
Ans. B \$66-66 $\frac{2}{3}$ and C \$33-33 $\frac{1}{3}$.

7. A merchant failing, owes to B £500 and to C £900 ; but has only £1100 to meet these demands. How much should each creditor receive? *Ans.* B £392 $\frac{2}{3}$ and C £707 $\frac{1}{3}$.
8. Three merchants load a ship with butter ; B gives 200 casks, C 300, and D 400 ; but when they are at sea it is found necessary to throw 180 casks overboard. How much of this loss should fall to the share of each merchant?
Ans. B should lose 40 casks, C 60, and D 80.
9. Three persons are to pay a tax of \$100, according to their estates. B's yearly property is \$800, C's \$600, and D's \$400. How much is each person's share?
Ans. B's \$44-44 $\frac{1}{4}$, C's \$33-33 $\frac{1}{4}$, and D's \$22-22 $\frac{1}{4}$.
10. Divide 120 into three such parts as shall be to each other as 1, 2, and 3. *Ans.* 20, 40, and 60.
11. A ship worth \$900 is entirely lost ; $\frac{1}{4}$ of it belonged to B, $\frac{1}{4}$ to C, and the rest to D. What should be the loss of each, \$540 being received as insurance?
Ans. B \$45, C \$90 and D \$225.
12. Three persons have gained \$1320 ; if B were to take \$6, C ought to take \$4, and D \$2. What is each person's share?
Ans. B's \$660, C's \$440, and D's \$220.
13. Three persons join ; B and C put in a certain stock, and D puts in £1090 ; they gain £110, of which B takes £35, and C £29. How much did B and C put in ; and D's share of the gain?
Ans. B put in £829 6s. 11 $\frac{1}{4}$ d.,
C " £687 3s. 5 $\frac{1}{4}$ d.,
and D's part of the profit is £46.

COMPOUND PARTNERSHIP.

52. When the partners employ their capital for different periods of time, the partnership is called Compound Partnership or Compound Fellowship.

It is likewise called Double Partnership, or Partnership With Time.

For example : Suppose A puts in \$200 for 3 years, and B \$300 for 4 years, and they make a certain gain or loss. This would give a case of Compound Partnership.

In such cases it is plain that each man's share of the profits depends upon two circumstances :

1st. The amount of his stock ; and

2nd. The period for which it is continued in the business.

Also that when the times are equal, the shares of the gain or loss are as the stocks ; when the stocks are equal, the shares are as the times ; and when neither the times nor the stocks are equal, the shares are as their products.

Hence, for Compound Partnership we have the following—

RULE.

Multiply each man's stock by the time he continues it in trade ; then say, as the sum of the products is to each particular product, so is the whole gain or loss to each man's share of the gain or loss.

EXAMPLE 1.—A contributes \$120 for 6 months, B \$336 for 11 months, and C \$384 for 8 months; and they lose \$56. What is C's share of the loss?

OPERATION.

$$\left. \begin{array}{l} \$120 \times 6 = \$720 \text{ for one month} \\ 336 \times 11 = 3696 \text{ for one month} \\ 384 \times 8 = 3072 \text{ for one month} \end{array} \right\} = \$7488 \text{ for one month.}$$

$$\$7488 : \$3072 :: \$56 : \text{C's share; or } \frac{\$3072 \times 56}{7488} = \$22.671.$$

EXPLANATION.—It is clear that \$120 contributed for 6 months are, as far as the gain or loss is concerned, the same as 6 times \$120, or \$720, contributed for one month. Hence A's contribution may be taken as \$720 for 1 month; and, for the same reason, B's as \$3696 for the same time; and C's as \$3072, also for the same time. This reduces the question to one in Simple Fellowship.

EXERCISE.

- Three merchants enter into partnership; B puts in \$357 for 5 months, C \$371 for 7 months, and D \$154 for 11 months; and they gain \$347.20. What should be each person's share of it? *Ans.* B's \$102, C's \$148.40, and D's \$96.80.
- B, C, and D pay \$100 as the year's rent of a pasture. B puts 40 cows on it for 6 months, C 30 for 5 months, and D 50 for the rest of the time. How much of the rent should each person pay? *Ans.* B \$87.27 $\frac{1}{3}$, C \$54.54 $\frac{2}{3}$, and D \$18.18 $\frac{1}{3}$.
- Three dealers, A, B, and C, enter into partnership, and in a certain time make £291 13s. 4d. A's stock, £150, was in trade 6 months; B's, £200, 3 months; and C's, £125, 16 months. What is each person's share of the gain? *Ans.* A's is £75, B's, £50, and C's, £166 13s. 4d.
- Three persons have received \$665 interest; B had put in \$4000 for 12 months, C \$3000 for 15 months, and D \$5000 for 8 months. How much is each person's part of the interest? *Ans.* B's \$240, C's \$225, and D's \$200.
- Three troops of horse rent a field, for which they pay \$320; the first sent into it 56 horses for 12 days, the second 64 for 15 days, and the third 80 for 18 days. What must each pay? *Ans.* The first must pay \$ 70,
The second " 100,
The third " 150.
- Three merchants are concerned in a steam-vessel; the first, A, puts in \$960 for 6 months; the second, B, a sum unknown for 12 months; and the third, C, \$640, for a time not known when the accounts were settled. A received \$1200 for his stock and profit, B \$2400 for his, and C \$1040 for his: what was B's stock, and C's time? *Ans.* B's stock was \$1600; and C's time was 15 months.

NOTE.—If A gain \$240 in 6 months, he would gain \$480 in 12 months; that is, A's stock and profit at the end of 12 months would be \$960 + \$480 = \$1440.

Then \$1440 : \$2400 :: \$960 : B's stock; or $\frac{2400 \times 960}{1440} = \1600 B's stock.

Again, B's stock : C's stock :: B's profit : C's profit for same time, viz: 12 months. That is \$1600 : \$640 :: \$800 : $\frac{640 \times 800}{1600} = \$320 =$ C's profit for 12 months.

Lastly, C's profit for 12 months : C's given profit :: 12 months : C's time; that is, \$320 : \$400 :: 12 months : $\frac{400 \times 12}{320} = 15$ mo. = C's time.

8. In the foregoing question A's gain was \$240 during 6 months, B's \$800 during 12 months, and C's \$400 during 15 months; and the sum of the products of their stocks and times is 34560. What were their stocks? *Ans.* A's was \$ 960,
B's " 1600,
C's " 640.
9. In the same question the sum of the stocks is \$3200; A's stock was in trade 6 months, B's 12 months, and C's 15 months; and at the settling of accounts, A is paid \$240 of the gain, B \$800, and C \$400. What was each person's stock? *Ans.* A's was \$960, B's \$1600, and C's \$640.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers following the questions refer to the Articles of the Section.

1. What is interest? (1)
 2. What is the meaning of the terms *per cent.* and *per annum*? (1)
 3. In what respect does interest differ from Commission and Brokerage? (2)
 4. What is the principal? (3)
 5. What is meant by the *rate per cent.*? (4)
 6. What is meant by the *rate per unit*? (5)
 7. What is the interest? (6)
 8. What is the amount? (7)
 9. Of how many kinds is interest? (8)
 10. Explain the distinction between Simple and Compound Interest. (9 and 10)
 11. In using formulas for interest, what is the meaning of the letters P, A, T, t, and r? (12)
 12. Deduce algebraically a full set of rules for Simple Interest. (12)
 13. How is the interest found when the *principal*, *rate per cent.*, and *time* are given? (13)
- NOTE.—Answer this and succeeding similar questions by giving the formula.
14. Interpret this formula. (13)
 15. When the *interest*, *rate per cent.*, and *time* are given, what is the rule for finding the principal? (14)
 16. Interpret this formula. (14)
 17. How is the rate per cent. found when the *interest*, *principal*, and *time* are given? (15)
 18. Interpret this formula. (15)
 19. When the *interest*, *principal*, and *rate* are given, how is the time found? (16)

20. Interpret this formula. (16)
21. When the *principal*, *rate*, and *time* are given, how is the amount found? (17)
22. Interpret this formula. (17)
23. When the *amount*, *rate*, and *time* are given, how do we find the *principal*? (18)
24. Interpret this formula. (18)
25. When the *amount*, *principal*, and *time* are given, how do we find the *rate*? (19)
26. Interpret this formula. (19)
27. When the *amount*, *principal*, and *rate* are given, how do we find the *time*? (20)
28. Interpret this formula. (20)
29. How do we find the time in which any sum of money will amount to any given number of times itself at a given rate? (21)
30. Interpret this formula. (21)
31. How do we find the rate at which any sum will amount to a given number of times itself in a given time? (22)
32. Interpret this formula. (22)
33. When the time and rate are given, how do we find to how many times itself a given sum will amount? (23)
34. Interpret this formula. (23)
35. How do we find the interest on \$1 at 6 per cent. per annum for any number of months? (24)
36. How do we find the interest on \$1 at 6 per cent. for any number of days? (25)
37. How do we find the interest of any sum for any given time at 6 per cent.? (26)
38. How may we find the interest at any other rate than 6 per cent.? (27)
39. How do we compute interest on notes, &c., when partial payments are made? (28)
40. What is the rule for calculating Compound Interest? (30)
41. How is Compound Interest calculated by the table given in Art. 31? (32)
42. How do we ascertain the present worth of a debt due some given time hence, allowing Compound Interest at a given rate? (33)
43. What is Discount? (34)
44. What is meant by the present worth of a debt, note, &c.? (35)
45. How do we compute the present worth of a debt or note? (36)
46. What is Bank Discount? (37)
47. What is the distinction between Bank Discount and True Discount? (38 and 35)
48. What are days of grace? (39)
49. What are the two kinds of notes discounted at banks? (40)
50. How do we calculate the bank discount on notes, &c.? (41)
51. How do we find what amount to put on the face of a note so that its present value shall be a certain sum? (42)
52. What is meant by the Equation of Payments? (43)
53. What is meant by the *mean time* or *equated time* of payment? (44)
54. How do we find the equated time of payment? (45)
55. What is Partnership or Fellowship? (46)
56. What are the persons associated together in partnership called? (47)
57. What is the money employed in the business called? (48)
58. What is meant by the dividend? (49)
59. What is the distinction between Simple and Compound Fellowship? (50 and 52)
60. By what other names is Simple Partnership known? (50)
61. What is the rule for Simple Partnership? (51)
62. What is the rule for Compound Partnership? (52)

SECTION IX.

PROFIT AND LOSS, BARTER, ALLIGATION, CURRENCIES, EXCHANGE, &c.

PROFIT AND LOSS.

1. Profit and Loss is a rule by which we are enabled to ascertain what we gain or lose in mercantile transactions. It also instructs us how much we must increase or diminish the price of our goods in order that our gain or loss may be so much per cent.

CASE I.

2. *To find the total gain or loss on a certain quantity of goods when the prime cost and selling price are given :*

FIRST RULE.

Find the price of the goods at prime cost and also at the selling price. The difference will be the whole gain or loss.

EXAMPLE 1.—What do I gain if I buy 207 cords of wood at \$3.78 per cord and sell it at \$4.25 ?

OPERATION.

207 cords @ \$4.25 = \$879.75 = whole sum for which goods were sold.
207 cords @ \$3.78 = \$782.46 = whole cost.

Difference = \$97.29 = whole gain = *Ans.*

EXAMPLE 2.—If I purchase 900 bushels of wheat at \$1.47 per bushel and sell it at \$1.25, what do I lose upon the whole transaction ?

OPERATION.

900 bushels @ \$1.47 = \$1323 = whole cost.
900 bushels @ \$1.25 = \$1125 = whole sum received for wheat.

\$198 = whole loss = *Ans.*

SECOND RULE.

Find the difference between the buying and selling price of a bushel, lb., yard, &c.

Multiply the gain or loss per bushel, lb., yard, &c., by the number of bushels, lbs., or yards, and the result will be the whole gain or loss.

EXAMPLE 3.—Bought 211 yards of flannel at $37\frac{1}{2}$ cents per yard, and sold it at 45 cents; required my total gain?

OPERATION.

\$0.375 = buying price.

\$0.45 = selling price.

—————

\$0.075 = gain per yard.

\$0.075 \times 211 = \$15.825. *Ans.*

NOTE.—This second rule affords the shorter method of finding the gain or loss.

EXERCISE.

- Bought 317 lbs. of butter at 9 cents per lb., and sold it at $12\frac{1}{2}$ cents; what was my gain on the whole? *Ans.* \$11.095.
- Bought 2138 bushels of potatoes at $87\frac{1}{2}$ cents per bushel, and sold them at \$1.20; what was my gain on the whole?
Ans. \$694.85.
- Bought 13 barrels of sugar, each weighing 317 lbs. net at 15 cents per lb., and sold the whole for \$735; how much did I gain or lose on the transaction? *Ans.* Gained \$116.85.
- Bought 17 kegs of wine, each containing 22 gallons, at \$3.15 per gallon, and paid in addition \$26.33 for carriage, &c., and an *ad valorem* duty of $37\frac{1}{2}$ per cent. I sold the whole for \$1625; what was my gain or loss? *Ans.* Loss \$21.2175.

CASE II.

3. Let it be required to find for what sum I must sell a house which cost \$2900 so that I may gain 15 per cent.

Here for every \$100 the house cost me I am to receive \$115, or for every \$1 cost I am to receive \$1.15.

The selling price must evidently be as many times \$1.15 as the buying price contains \$1; i. e., $\$1.15 \times 2900 = \3335.00 . *Ans.*

Again: If a person buys a horse for \$230, and afterwards sells it so as to lose 11 per cent.; how much does he receive for it?

Here for every \$1 he paid for the horse he receives only \$0.89 (since he loses 11 per cent., i. e., 11 cents on the \$1.)

Then, the selling price will obviously be $\$0.89 \times 230 = \204.70 . *Ans.*

Hence, to find at what price an article must be sold so as to gain or lose a specified per centage, the cost price being given:—

RULE.

Find (Art. 2, Sect. VII.) how much must be received for each dollar of the buying price, and multiply this by the whole buying price. The result will be the selling price.

EXAMPLE 8.—Bought a quantity of oatmeal for \$1793.80. For what must I sell it so as to gain 8 per cent.?

OPERATION.

Here for every \$1 I expend I desire to receive \$1.08; hence, the selling price will be $\$1.08 \times 1793.80 = \1937.304 . *Ans.*

EXAMPLE 9.—Bought a lot of sheep for \$7000, and am willing to lose 3 per cent. For what sum must I sell?

OPERATION.

Here for every \$1 I expend I am willing to receive \$0.97, and hence the selling price will be $\$0.97 \times 7000 = \6790 . *Ans.*

EXERCISE.

10. Bought cordwood at \$3.25 per cord; at what rate per cord must I sell it in order to gain 30 per cent. ? *Ans.* \$4.22½.
11. Bought a stock of goods for \$13420; for how much must it be sold in order to gain 5 per cent. ? *Ans.* \$14091.
12. Bought a quantity of wool at 11 cts. a lb., and wish to sell so as to gain 15 per cent.; at what rate per lb. must I sell it ? *Ans.* 12½³/₁₀ cents.
13. Bought axes at \$15.25 a doz., and desire to sell them so as to gain 23 per cent.; at what rate per doz. must I sell ? *Ans.* \$18.75½.
14. Bought a farm for \$7890, and am willing to lose 11 per cent.; at what price must I sell ? *Ans.* \$7022.10.

CASE III.

4. Let it be required to find what per cent. of profit a merchant makes by buying tea at 43 cents per lb. and selling it at 67 cts.

Here the gain on each lb. is 24 cents.

That is every 43 cents invested gives a gain of 24 cents.

Therefore every cent invested gains $\frac{1}{43}$ of 24 cents = $\frac{24}{43}$ cents.

And hence, the gain per cent. = $\frac{24}{43} \times 100 = \frac{2400}{43} = 55.8$ per cent.

Hence to find the rate per cent. of profit or loss when the prime cost and selling price are given, we have the following:—

RULE.

Find the difference between the buying and selling price, and hence the gain or loss per unit.

Multiply this by 100, and the result will be the gain or loss per cent.

EXAMPLE 15.—A speculator invests \$44400 in stocks, and sells out for \$50000; what per cent. does he make by the operation?

OPERATION.

Here the whole gain is $\$50000 - \$44400 = \$5600$.

That is \$44400 gain \$5600, and therefore \$1 gains $\frac{5600}{44400} = \frac{14}{111}$ of a dollar.

Hence gain per cent. = $\frac{14}{111} \times 100 = \frac{1400}{111} = 12.6$. *Ans.*

NOTE.—The above and all similar questions may be solved by Proportion. Thus this question is, if \$44400 gain \$5600, what will \$100 gain?

And the statement is $\$44400 : \$100 :: \$5600 : \text{Ans.} = \frac{5600 \times 100}{44400} = 12.6$.

EXERCISE.

16. Bought tea at 60 cents a lb., and sold it at $87\frac{1}{2}$ a lb.; how much did I gain per cent. ? *Ans.* $45\frac{5}{8}$.
17. Bought coffee at 13 cents and sold it at 11 cents a pound; what was my loss per cent. ? *Ans.* $15\frac{2}{3}$.
18. Bought flour at \$6.20 a barrel, and sold it at \$7.80; what was the per cent. of profit ? *Ans.* $25\frac{1}{3}$ per cent.
19. Bought cloth at \$2.75 per yard, and sold it at \$3.10; what was my gain per cent. ? *Ans.* $12\frac{2}{3}$ per cent.
20. Bought oats at \$0.47 per bushel, and sold them at \$0.56; what was my gain per cent. ? *Ans.* $19\frac{1}{4}$ per cent.
21. Bought meat at 12 cents per lb., and sold it at $10\frac{1}{2}$ cents a pound; what was my loss per cent. ? *Ans.* $12\frac{1}{2}$ per cent.
22. Bought a horse for \$93, and sold it for \$127: what per cent. of profit did I make ? *Ans.* $36\frac{2}{3}$.
23. A man bought a farm for \$6742.50, and sold it for \$6000; what was his loss per cent. ? *Ans.* $11\frac{1}{10}$ per cent.
24. If I purchase a house for \$5700, a horse for \$275, and pay \$1987.32 for household furniture and a carriage, and then sell the whole for \$8750, what is my gain or loss per cent. ? *Ans.* Gain 9.89 or nearly 10 per cent.
25. I purchase 723 yards of black silk velvet in Paris and pay \$4.25 a yard; I further pay 7 per cent. for insurance, \$23.70 for carriage, \$2.70 for harbor dues, \$3.16 for wharfage and storage, and an *ad valorem* duty of 22 per cent., and then sell the whole for \$5270; what is my gain or loss per cent. ? *Ans.* Gain 31.96749 or nearly 32 per cent.

CASE IV.

5. Let it be required to find the prime cost of cloth which I sold for \$4 and gained 10 per cent. thereby.

Here the gain on \$1 was 10 cents, or what I sold for \$1.10 cost me only \$1.

Therefore the cost price will contain \$1 as many times as the selling price contains \$1.10.

That is the cost price = $\frac{1}{1.10} = \$3.636$. *Ans.*

Hence, to find the cost price, the selling price and the gain or loss per cent. being given, we have the following:—

RULE.

Find the gain or loss per unit, and add it to unity if it be gain but subtract it from unity if it be loss.

Divide the selling price by the quantity thus obtained, and the result will be the cost price.

Or say as \$100 + gain per cent. (or as \$100—loss per cent.) is to \$100 so is the selling price to the cost price.

EXAMPLE 26.—Sold a quantity of coal for \$719, and lost 7 per cent. by the transaction; what was the prime cost?

OPERATION.

1ST RULE.—Loss on \$1 is 7 cents, or for every \$1 paid I receive \$0.93.
Hence cost = $\$1 \frac{1}{.93} = \773.117 .

2ND RULE.—\$.93 : \$100 :: \$719 : *Ans.* = $\frac{100 \times 719}{93} = \773.118 .

EXERCISE.

27. For what did I buy a quantity of sugar which I sold for \$24.60, losing 4 per cent.?
Ans. \$25.625.
28. A gentleman sold his library for \$2360, which was 10 per cent. less than cost; what did he give for it? *Ans.* \$2622.22.
29. A farmer sold his farm for \$7400, gaining 11 per cent. on the prime cost; what did he give for it? *Ans.* \$6666.666.
30. A merchant sold a quantity of silk velvet for \$3789.40, gaining 17 per cent. by the transaction; required the buying price? *Ans.* \$3238.803.
31. Sold a lot of cattle for \$2740, losing 13 per cent. by the transaction; what did I give for them? *Ans.* \$3149.425.

BARTER.

6. Barter signifies an exchange of goods or articles of commerce at prices agreed upon so that neither party in the transaction may sustain loss.

7. *The principle of solution depends upon finding the value of the commodity whose price and quantity are given, and thence the equivalent quantity of a second commodity of a given price, or the equivalent price of a given quantity of a second commodity.*

EXAMPLE 1.—How much tea at \$1.10 per lb. ought to be given for 712 lb. of sugar at 13 cents per lb.?

OPERATION.

712 lbs. of sugar at 13 cents per lb. = \$92.56, and $\$92.56 \div \$1.10 = 84.1454$ lbs. = 84 lbs. 2½ oz. *Ans.*

EXAMPLE 2.—I desire to barter 96 lbs. of sugar, which cost me 8 cents per lb., but which I sell at 13 cents, giving 9 months' credit, for calico which another merchant sells for 17 cents per yard, giving 6 months' credit. How much calico ought I to receive?

OPERATION.

I first find at what price I could sell my sugar, were I to give the same credit as he does—

If 9 months give me 5 cents profit, what ought 6 months to give?

$$9 : 6 :: 5 : \frac{6 \times 5}{9} = \frac{30}{9} = 3\frac{1}{3} \text{ cents.}$$

Hence, were I to give 6 months' credit, I should charge $8 + 3\frac{1}{3} = 11\frac{1}{3}$ cents per lb. Next—

As my selling price is to my buying price, so ought his selling to be to his buying price, both giving the same credit.

$$11\frac{1}{2} : 8 :: 17 : \frac{8 \times 17}{11\frac{1}{2}} = 12 \text{ cents.}$$

The price of my sugar, therefore, is 96×8 cents, or \$7.68: and of his calico, 12 cents per yard.

Hence $\frac{\$7.68}{12} = 64$, is the required number of yards.

EXERCISE.

3. A has coffee which he barter at 10 cents the lb. more than it cost him, against tea which stands B in \$2, but which he rates at \$2.50 per lb. How much did the coffee cost at first?

Ans. 40 cents.

4. A has silk which cost \$2.80 per lb.; B has cloth at \$2.50, which cost only \$2 the yard. How much must A charge for his silk, to make his profit equal to that of B?

Ans. \$3.50.

I have cloth at 8 cents the yard, and in barter charge for it 13 cents, and give 9 months' time for payment; another merchant has goods which cost him 12 cents per lb., and with which he gives 6 months' time for payment. How high must he charge his goods to make an equal barter?

Ans. At 17 cents.

6. K and L barter. K has cloth worth \$1.60 the yard, which he barter at \$1.85 with L, for linen cloth at 60 cents per yard, which is worth only 55 cents. Who has the advantage; and how much linen does L give to K for 70 yards of his cloth? *Ans.* L gives K $215\frac{1}{2}$ yards; and K has the advantage.

7. B has five tons of butter, at \$102 per ton, and $10\frac{1}{2}$ tons of tallow, at \$135 per ton, which he barter with C; agreeing to receive \$600.30 in ready money, and the rest in beef at \$4.20 per barrel. How many barrels is he to receive?

Ans. 316.

ALLIGATION.

8. Alligation is the method of finding the value of a mixture of ingredients of different values, or of forming a compound which shall have a given value.

NOTE.—The term *alligation* is derived from the Latin word *alligo* "to tie or bind," the reference being to the manner of connecting or tying the numbers together in a certain class of questions.

9. Alligation is divided into *Alligation Medial* and *Alligation Alternate*.

10. Alligation Medial (Latin *medius*, "mean or average") enables us to find the value of a mixture when the

ingredients, of which it is composed and their prices are known.

11. Alligation Alternate enables us to find what proportion must be taken of several ingredients, whose prices are known, in order to form a compound of a given price.

ALLIGATION MEDIAL.

12. Let it be required to find the price per lb. of a mixture containing 47 lbs. of sugar at 11 cents per lb., 29 lbs. at 13 cents, and 24 lbs. at 17 cents.

OPERATION.

47 lbs. at 11 cents = 517 cents.
 29 lbs. at 13 cents = 377 cents.
 24 lbs. at 17 cents = 408 cents.

Then 100 lbs. cost 1302 cents and 1 lb. will cost $\frac{1302}{100} = 13\frac{1}{10}$ cents.

Hence for Alligation Medial we deduce the following:—

RULE.

Divide the entire cost of the whole mixture by the sum of the ingredients, and the quotient will be the price per unit of the mixture.

EXAMPLE 1.—What will be the price per lb. of a mixture of tea containing 7 lbs. at \$0.50 per lb., 11 lb. at \$0.80, 19 at \$1.06, and 3 lbs. at \$1.23?

OPERATION.

| | | | | |
|--------|---|--------|---|---------|
| 7 lbs. | @ | \$0.50 | = | \$3.50 |
| 11 " | @ | \$0.80 | = | \$8.80 |
| 19 " | @ | \$1.06 | = | \$20.14 |
| 3 " | @ | \$1.23 | = | \$3.69 |

40 lbs. = sum of ingredients. \$36.13 = Total cost.

40) \$36.13 (\$0.90 $\frac{13}{10}$). Ans.

36.0
 ———
 .13

EXAMPLE 2.—A goldsmith has 3 lbs. of gold 22 carats fine, and 2 lbs. 21 carats fine. What will be the fineness of the mixture?

In this case the value of each kind of ingredient is represented by a number of carats—

OPERATION.

| | | |
|-------------|-------|-----------|
| 3 lbs. × 22 | = | 66 carats |
| 2 " × 21 | = | 42 " |
| 5 | 5)108 | " |

The mixture is $21\frac{3}{5}$ carats fine.

EXERCISE.

3. Having melted together 7 oz. of gold 23 carats fine, $12\frac{1}{2}$ oz. 21 carats fine, and 17 oz. 9 carats fine, I wish to know the fineness of each ounce of the mixture? *Ans.* $15\frac{1}{3}$ carats.
4. A vintner mixed 2 gallons of wine, at 14s. per gallon, with 1 gallon at 12s., 2 gallons at 9s., and 4 gallons at 8s. What is one gallon of the mixture worth? *Ans.* 10s.
5. A farmer mixes 15 bushels of wheat worth \$1.20 with 30 bushels worth \$1.50, and 60 bushels worth \$1.10 and 83 bushels worth \$1.75. What is one bushel of the mixture worth? *Ans.* \$1.458.
6. A grocer mixes together 12 lbs. of tea at 50 cents, 16 lbs. at 72 cents, 22 lbs. at 65 cents, 18 lbs. at 85 cents, and 100 lbs. at 42 cents. How much per lb. is the mixture worth? *Ans.* $53\frac{1}{2}$ cents.

ALLIGATION ALTERNATE.

13. Alligation Alternate is the reverse of Alligation Medial, and may be proved by it.

CASE I.

14. Given the prices of the ingredients, to find the proportion in which they must be mixed in order that the compound may be worth a given price :—

RULE.

Set down the prices of the ingredients in two columns, placing those greater than the price of the compound to the left, and those less than it to the right.

Between these columns form two others composed of the differences between the prices of the several ingredients and of the compound; writing each difference next to the number by which it was obtained.

Link, by means of a line, the left-hand differences to the right-hand differences in any order.

Then each difference will express how much of the quantity with whose difference it is connected, should be taken to form the required mixture.

If any difference is connected with more than one other difference, it is to be considered as repeated for each of the differences with which it is connected; and the sum of the differences with which it is connected is to be taken as the required amount of the quantity whose difference it is.

EXAMPLE 7.—How many pounds of tea at 5s. and 8s. per lb., would form a mixture worth 7s. per lb.?

OPERATION.

Prices. Differences. Prices.

$$7 = 8 - 1 \quad \overbrace{\quad\quad}^2 + 5 = 7$$

1 is connected with 2s., the difference between the 7, the required price, and 5s.: hence there must be 1 lb. at 5s. 2 is connected with 1, the difference between 8s. and the required price; hence there must be 2 lbs. at 8s. Then 1 lb. of tea at 5s. and 2 lbs. at 8s. per lb., will form a mixture worth 7s. per lb.—as may be proved by the last rule.

It is evident that any equimultiples of these quantities would answer equally as well; hence a great number of answers may be given to such a question.

EXAMPLE 8.—How much sugar at 9d., 7d., 5d., and 10d., will produce sugar at 8d. per lb.?

OPERATION.

Prices. Differences. Prices.

$$8 = \left\{ \begin{array}{l} 9 - 1 \quad \overbrace{\quad\quad}^1 + 7 \\ 10 - 2 \quad \overbrace{\quad\quad}^3 + 5 \end{array} \right\} = 8$$

1 is connected with 1, the difference between 7d. and the mean, 8; hence there is to be 1 lb. of sugar at 7d. per lb. 2 is connected with 3, the difference between 5d. and the mean; hence there is to be 2 lbs. at 5d. 1 is connected with 1, the difference between 9d. and the mean; hence there is to be 1 lb. at 9d. And 3 is connected with 2, the difference between 10d. and the mean; hence there are to be 3 lbs. at 10d. per lb.

Consequently we are to take 1 lb. at 7d. and 2 lbs. at 5d., 1 lb. at 9d. and 3 lbs. at 10d. If we examine the price of the mixture these will give (Art. 12), we shall find it to be the given mean.

EXAMPLE 9.—What quantities of tea at 4s., 6s., 8s., and 9s. per lb., will produce a mixture worth 5s.?

OPERATION.

Prices. Differences. Prices.

$$5 = \left\{ \begin{array}{l} 8 - 3 \quad \overbrace{\quad\quad}^1 + 4 \\ 6 - 1 \\ 9 - 4 \end{array} \right\} = 5$$

3, 1, and 4 are connected with 1s., the difference between 4s. and the mean; therefore we are to take 3 lbs. + 1 lb. + 4 lbs. of tea, at 4s. per lb. 1 is connected with 3s., 1s., and 4s., the differences between 8s., 6s., and 9s., and the mean; therefore we are to take 1 lb. of tea at 8s., 1 lb. of tea at 6s., and 1 lb. at 9s. per lb.

EXAMPLE 10.—How much of any thing at 3s., 4s., 5s., 7s., 8s., 9s., 11s., and 12s. per lb., would form a mixture worth 6s. per lb.?

OPERATION.

Prices. Differences. Prices.

$$6 = \left\{ \begin{array}{l} 7 - 1 \quad \overbrace{\quad\quad}^3 + 3 \\ 8 - 2 \quad \overbrace{\quad\quad}^2 + 4 \\ 9 - 3 \quad \overbrace{\quad\quad}^1 + 5 \\ 11 - 5 \\ 12 - 6 \end{array} \right\} = 6$$

1 lb. at 3s., 2 lbs. at 4s., 3 lbs. at 7s., 2 lbs. at 8s., 3 + 5 + 6; i.e., 14 lbs. at 5s., 1 lb. at 6s., 1 lb. at 11s., and 1 lb. at 12s. per lb. will form the required mixture.

NOTE.—The principle upon which this rule proceeds is that the excess of one ingredient above the mean is made to counterbalance what the other wants of being equal to the mean. Thus in example 7, 1 lb. at 6s. per lb. gives a *deficiency* of 2s.; but this is corrected by 2s. *excess* in the 2 lbs. at 8s. per lb.

In example 8, 1 lb. at 7d. gives a *deficiency* of 1d., 1 lb. at 9d. gives an *excess* of 1d.; but the excess of 1d. and the deficiency of 1d. exactly neutralize each other.

Again, it is evident that 2 lbs. at 5d. and 3 lbs. at 10d. are worth just as much as 5 lbs. at 8d.—that is, 8d. will be the average price if we mix 2 lbs. at 5d. with 3 lbs. at 10d.

EXERCISE.

- How much wheat at \$1.60, \$1.40, \$1.10, and \$1 per bushel must be mixed together in order to form a mixture worth \$1.25 per bushel? Give at least *two* sets of answers.
Ans. 35 bushels at \$1.10, 15 at \$1.60, 15 at \$1.00, and 25 at \$1.40.
35 bushels at \$1.00, 15 at \$1.40, 15 at \$1.10, and 25 at \$1.60.
- How much wine at 60 cents, 50 cents, 42 cents, 38 cents, and 30 cents per quart, will make a mixture worth 45 cents a quart? *Ans.* 15 qts. at 42 c., 5 qts. at 30 c., 3 qts. at 60 c. and 22 qts. at 50 c.
- A merchant has sugar worth 10 cents, 12 cents, 14 cents, 15 cents, 16 cents, 17 cents, and 18 cents per pound, and wishes to form a mixture worth $12\frac{1}{2}$ cents a lb. How many pounds of each must he use? *Ans.* $2\frac{1}{2}$ lbs. at 14 c., $1\frac{1}{2}$ lbs. at 10 c., 16 lbs. at 12 c. and $\frac{1}{2}$ lb. at each other price.
- A grocer has sugar at 5d., 7d., 12d., and 13d. per lb. How much of each kind will form a mixture worth 10d. per lb? *Ans.* 2 lbs. at 5d., 3 lbs. at 7d., 5 lbs. at 12d., and 3 lbs. at 13d.

CASE II.

15. When a given quantity of one of the ingredients is to be taken :—

I. Find the proportional quantities of the ingredients as in Case I.

II. Then say, as the amount of the ingredient as thus found is to the given amount of the same ingredient, so is the amount of any other ingredient (found by Case I.) to the required quantity of that other.

EXAMPLE 15.—29 lbs. of tea at 4s. per lb., is to be mixed with teas at 6s., 8s., and 9s. per lb., so as to produce what will be worth 5s. per lb. What quantities must be used?

OPERATION.

By Case I we find that 8 lbs. of tea at 4s., and 1 lb. at 6s., 1 lb. at 8s., and 1 lb. at 9s., will make a mixture worth 5s. per lb.

Therefore 8 lbs. (the quantity of tea at 4s. per lb., as found by the rule) : 29 lbs. (the given quantity of the same tea) :: 1 lb. (the quantity of tea at 6s. per lb., as found by the rule;) or $\frac{1 \times 29}{8}$ lb. = $3\frac{5}{8}$ lbs. *Ans.*

We may in the same manner find what quantities of tea at 8s. and 9s. per lb., correspond with 29 lb. of tea at 4s. per lb.

EXAMPLE 16.—A refiner has 10 oz. of gold 20 carats fine, and melts it with 16 oz. 18 carats fine. What must be added to make the mixture 22 carats fine?

$$10 \text{ oz. of 20 carats fine} = 10 \times 20 = 200 \text{ carats.}$$

$$16 \text{ oz. of 18 carats fine} = 16 \times 18 = 288$$

$$\hline 26 : 1 :: 488 : 18\frac{1}{3} \text{ carats, the fine-}$$

ness of the mixture.

24—22 = 2 carats baser metal, in a mixture 22 carats fine.

24— $18\frac{1}{3}$ = $5\frac{2}{3}$ carats baser metal in a mixture $18\frac{1}{3}$ carats fine.

Then 2 carats : 22 carats :: $5\frac{2}{3}$: $57\frac{7}{3}$ carats of pure gold—required to change $5\frac{2}{3}$ carats baser metal into a mixture 22 carats fine. But there are already in the mixture $18\frac{1}{3}$ carats gold; therefore $57\frac{7}{3}$ — $18\frac{1}{3}$ = $38\frac{2}{3}$ carats gold are to be added to every ounce. There are 26 oz.; therefore $26 \times 38\frac{2}{3}$ = 1008 carats of gold are wanting. There are 24 carats in every oz.; therefore $\frac{1008}{24}$ carats = 42 oz. of gold must be added. There will then be a mixture containing

| oz. | car. | car. |
|-----------|------|-------|
| 10 × 20 = | | 200 |
| 16 × 18 = | | 288 |
| 42 × 24 = | | 1008 |
| <hr/> | | <hr/> |

68 : 1 oz. :: 1496 : 22 carats, the required fineness.

EXERCISE.

17. How much molasses at 16 cents, at 19 cents, and at 23 cents per quart must be mixed with 87 quarts at 31 cents in order that the mixture may be worth 25 cents per quart?
Ans. $30\frac{2}{3}$ qts. at each price.
18. How much oats at 37 cents per bushel and barley at 68 cts. per bushel must be mixed with 70 bushels of peas at 80 cts. a bushel so that the mixture may be worth 75 cents per bushel?
Ans. $7\frac{1}{3}$ bush. at each price.
19. How much brass at 14d. per lb., and pewter at 10½d. per lb., must I melt with 50 lbs. of copper at 16d. per lb., so as to make the mixture worth 1s. per lb.?
Ans. 50 lbs. of brass, and 200 lbs. of pewter.
20. How much gold of 21 and 23 carats fine must be mixed with 30 oz. of 20 carats fine, so that the mixture may be 22 carats fine?
Ans. 30 of 21, and 90 of 23.

CASE III.

16. When the quantity of the compound is given as well as the price—

1. Find the proportional quantities as in Case I.

II. Then say, as the sum of the proportional quantities is to each proportional quantity, so is the given quantity to the corresponding part of each.

EXAMPLE 21.—What must be the amount of tea at 4s. per lb. in 736 lb. of a mixture worth 5s. per lb., and containing tea at 6s., 8s., and 9s., per lb. ?

To produce a mixture worth 5s. per lb., we require 8 lb. at 4s., 1 at 8s., 1 at 6s., and 1 at 9s. per lb. (Art. 14). But all of these added together, will make 11 lbs. in which there are 8 lbs. at 4s. Therefore

lbs. lbs. lbs. lbs. lbs. oz.

$$11 : 8 :: 736 : \frac{8 \times 736}{11} = 535 \frac{4}{11}, \text{ the required quantity of tea at 4s.}$$

That is, in 736 lbs. of the mixture there will be 535 lbs. $\frac{4}{11}$ oz. at 4s. per lb. The amount of each of the other ingredients may be found in the same way.

EXERCISE.

22. A druggist is desirous of producing, from medicine at \$1·00, \$1·20, \$1·60, and \$1·80 per lb., 168 lbs. of a mixture worth \$1·40 per lb.; how much of each kind must he use for the purpose? *Ans.* 28 lbs. at \$1·00, 56 lbs. at \$1·20, 56 lbs. at \$1·60, and 28 lbs. at \$1·80 per lb.
23. 27 lbs. of a mixture worth 4s. 4d. per lb. are required. It is to contain tea at 5s. and at 3s. 6d. per lb.; how much of each must be used? *Ans.* 15 lbs. at 5s., and 12 lbs. at 3s. 6d.
24. How much brandy at \$2·40, \$2·60, \$2·80, and \$2·90, per gallon, must there be in one hogshead of a mixture worth \$2·70, per gallon? *Ans.* 18 gals. at \$2·40, 9 gals. at \$2·60, 9 gals. at \$2·80, and 27 gals. at \$2·90 per gallon.

EXCHANGE OF CURRENCIES.

17. Exchange of Currencies is the process of changing a sum of money expressed in the denominations of one country to an equivalent sum expressed in the denominations of another country.

18. By the *currency of a country* is meant the *coins, or money, or circulating medium* of trade of that country.

19. The *intrinsic value* of a coin is determined by the *kind, purity, and quantity* of metal it contains.

20. The *relative value or commercial value* of a coin is its market value, and is fixed by law and commercial usage.

FOREIGN MONEYS OF ACCOUNT,

WITH THE PAR VALUE OF THE UNIT, AS FIXED BY COMMERCIAL USAGE, EXPRESSED IN DOLLARS AND CENTS.

| | |
|---|---------|
| AUSTRIA.—60 kreutzers=1 florin; 1 florin (silver)=..... | \$0'485 |
| BELGIUM.—100 cents=1 guilder or florin; 1 guilder (silver)=..... | '40 |
| BRAZIL.—1000 rees=1 milree=..... | '828 |
| BREMEN.—5 schwares=1 grote; 72 grotes=1 rix-dollar (silver)=.... | '787 |
| BRITISH INDIA.—12 pice=1 anna; 16 annas=1 Company's* rupee= | '445 |
| BUENOS AYRES.—8 rials=1 dollar currency (variable), mean value= | '93 |
| CANTON.—10 cash †=1 candarines; 10 cand.=1 mace, 10 mace=1 tael= | 1'48 |
| CAPE OF GOOD HOPE.—6 stivers=1 schiling; 8 schilings=1 rix-dollar=..... | '313 |
| CEYLON.—4 pice=1 fanam; 12 fanams=1 rix dollar=..... | '40 |
| CUBA, COLOMBIA, AND CHILI.—8 rials=1 dollar=..... | 1'00 |
| DENMARK.—12 pfeunings=1 skilling; 16 skillings=1 marc; 6 marcs=1 rix dollar=..... | '52 |
| ENGLAND.—4 farthings=1 penny; 12 pence=1 shilling; 20 shil.=£1= | 4'868 |
| FRANCE.—10 centimes=1 decime; 10 decimes=1 franc=..... | '186 |
| GREECE.—100 lepta=1 drachme; 1 drachme (silver)=..... | '168 |
| HOLLAND.—100 cents=1 florin or guilder; 1 florin (silver)=..... | '40 |
| HAMBURGH.—12 pfennings=1 schiling; 16 schil.=1 marc; 3 marcs=1 rix-dollar=..... | '84 |
| MALTA.—20 grains=1 taro; 12 tari=1 scudo; 2½ scudi=1 pezza=... | 1'00 |
| MILAN.—12 denari=1 soldo; 20 soldi=1 lira=..... | '20 |
| MEXICO.—8 rials=1 dollar=..... | 1'00 |
| MONTE VIDEO.—100 centesimos=1 rial, 8 rials=1 dollar=..... | '833 |
| NAPLES.—10 grani=1 carlino; 10 carlini=1 ducat (silver)=..... | '80 |
| NORWAY.—120 skillings=1 rix-dollar specie (silver)=..... | 1'06 |
| PAPAL STATES.—10 bajocchi=1 paolo; 10 paoli=1 scudo or crown= | 1'00 |
| PERU.—8 rials=1 dollar (silver)=..... | 1'00 |
| PORTUGAL.—400 rees=1 cruzado; 1000 rees=1 milree or crown=... | 1'12 |
| PRUSSIA.—12 pfennings=1 grosch (silver); 30 groschen=1 thaler or dollar=..... | '69 |
| RUSSIA.—100 copecks=1 ruble (silver)=..... | '78 |
| SARDINIA.—100 centesimi=1 lira=..... | '186 |
| SWEDEN.—48 skillings=1 rix-dollar specie=..... | 1'06 |
| SICILY.—20 grani=1 taro; 30 tari=1 oncia (gold)=..... | 2'40 |
| SPAIN.—3¼ maravedis=1 real of old plate ‡=..... | '10 |
| 8 reals=1 piastre; 4 piastres=1 pistole of exchange. | |
| 20 reals vellon=1 Spanish dollar =..... | 1'00 |

* The current silver rupee of Bombay, Madras, and Bengal, is worth \$0'444. In India also they use *cowries* for coin. These are small shells found in the Maldives and elsewhere; 2560 cowries make a *rupee*, and 100000 *rupees* make a *lac*.

† The cash, made of copper and lead, is said to be the only money coined in China.

‡ The old plate real is not a coin, but is the denomination in which exchanges are usually made.

| | |
|--|---------|
| ST. DOMINGO.—100 centimes = 1 dollar = | \$0'333 |
| TUSCANY.—12 denari di pezza = 1 soldo di pezza; 2 soldi di pezza = 1 pezza of 8 rials, 1 pezza (silver) = | '90 |
| TURKEY.—3 aspers = 1 para; 40 paras = 1 piastre (variable) about... | '096 |
| VENICE.—100 centesimi = 1 lira = | '186 |
| UNITED STATES OF AMERICA.—10 mills = 1 cent; 10 cents = 1 dime; 10 dimes = 1 dollar = | 1'00 |

21. *The following table exhibits the commercial value of the Foreign coins most frequently met with :*

| | |
|--|--------|
| GUINEA | \$5'00 |
| SOVEREIGN of Great Britain | 4'867 |
| CROWN of England | 1'216 |
| HALF-CROWN of England | '608 |
| SHILLING of England | '24½ |
| DOLLAR of the United States | 1'00 |
| FRANC of France | '18½ |
| FIVE FRANC PIECE of France | '93 |
| LIVRE TOURNOIS of France | '18½ |
| FORTY-FRANC PIECE of France | 7'66 |
| CROWN of France | 1'06 |
| LOUIS-D'OR of France | 4'66 |
| FLORIN of the Netherlands | '40 |
| GULDER of the Netherlands | '40 |
| FLORIN of Southern Germany | '40 |
| THALER or Rix-Dollar of Prussia and Northern Germany | '69 |
| RIX-DOLLAR of Bremen | '78¼ |
| FLORIN of Prussia | '22¾ |
| MARC-BANCO of Hamburg | '35 |
| FLORIN of Austria and city of Augsburg | '48½ |
| FLORIN of Saxony, Bohemia, and Trieste | '48 |
| FLORIN of Nuremburg, Frankfort, and Creveld | '40 |
| RIX-DOLLAR of Denmark | 1'00 |
| SPECIE-DOLLAR of Denmark | 1'05 |
| DOLLAR of Sweden and Norway | 1'06 |
| MILREE of Portugal | 1'12 |
| MILREE of Madeira | 1'00 |
| MILREE of Azores | '83½ |
| REAL-VELLON of Spain | '05 |
| REAL-PLATE of Spain | '10 |
| PISTOLE of Spain | 3'97 |
| RIAL of Spain | '12 |
| PISTEREEN | '18 |
| CROSS PISTAREEN | '16 |
| RUBLE (silver) of Russia | '76 |
| IMPERIAL of Russia | 7'83 |

| | |
|-----------------------------------|-------------------|
| DOUBLOON of Mexico..... | \$15'60 |
| HALF-JOE of Portugal..... | 8'53 |
| LIRA of Tuscany and Lombardy..... | '16 |
| LIRA of Sardinia..... | '18 $\frac{3}{4}$ |
| OUNCE of Sicily..... | 2'40 |
| DUCAT of Naples..... | '80 |
| CROWN of Tuscany..... | 1'05 |
| Florence LIVRE..... | '15 |
| Genoa "..... | '18 $\frac{3}{4}$ |
| Geneva "..... | '21 |
| Leghorn DOLLAR..... | '90 |
| Swiss LIVRE..... | '27 |
| SCUDO of Malta..... | '40 |
| Turkish PIASTRE..... | '05 |
| PAGODA of India..... | 1'84 |
| RUPEE of India..... | '44 $\frac{1}{2}$ |
| TAEL of China..... | 1'48 |

22. In Canada all accounts were kept in pounds, shillings, pence, and farthings, previous to the adoption of the decimal coinage by Act of Provincial Parliament in 1858. In the United States also accounts were similarly kept prior to the adoption of Federal Money in 1786. In the States, at the time Federal money was adopted, the *Colonial currency or bills of credit* had become more or less depreciated in value, i. e., a colonial shilling was worth less than a shilling sterling, &c., and the depreciation in value being greater in the currencies of some colonies than in others gave rise to the *different values* of the present old currencies of the different States.

TABLE OF CURRENCIES

IN CANADA AND THE UNITED STATES.

| | | |
|--|---------------|-----------------------|
| In Canada, Nova Scotia, New Brunswick, &c., | \$1 = 5s. | or £ $\frac{1}{4}$. |
| In N. Y., N. C., Ohio, and Mich., | \$1 = 8s. | or £ $\frac{2}{5}$. |
| In N. Eng., Va., Ky., Ten., Ia., Ill., Miss., Missouri, | \$1 = 6s. | or £ $\frac{3}{10}$. |
| In Penn., New Jer., Del., and Md., | \$1 = 7s. 6d. | or £ $\frac{3}{8}$. |
| In Georgia and S. C., | \$1 = 4s. 8d. | or £ $\frac{1}{3}$. |

NOTE.—The remaining States use the Federal money exclusively.

23. To reduce dollars and cents to old Canadian Currency, or to any State Currency:—

RULE.

Multiply the given sum by the value of \$1 in the required currency expressed as a fraction of a pound. The product will be pounds and decimals of a pound.

Reduce (Art. 58, Sect. IV.) the decimal to shillings, pence, and farthings.

EXAMPLE 1.—Reduce \$493·72 to Old Canadian Currency.

OPERATION.

$$493\cdot72 \times \frac{1}{4} = £123\cdot43 = £123 \text{ 8s. } 7\frac{1}{2}\text{d. } \textit{Ans.}$$

EXAMPLE 2.—Reduce \$749·80 to New England Currency.

OPERATION.

$$749\cdot80 \times \frac{3}{10} = £224\cdot94 = £222 \text{ 18s. } 9\frac{1}{10}\text{d. } \textit{Ans.}$$

EXAMPLE 3.—Reduce \$1111·11 to New York Currency.

OPERATION.

$$1111\cdot11 \times \frac{3}{8} = £444\cdot444 = £444 \text{ 8s. } 10\frac{1}{2}\frac{1}{4}\text{d. } \textit{Ans.}$$

EXERCISE.

4. Reduce \$1974·80 to New Jersey Currency. *Ans.* £740 11s.
5. Reduce \$765·43 to Michigan Currency. *Ans.* £306 3s. 5 $\frac{1}{2}$ d.
6. Reduce \$8172·19 to Old Canadian Currency.
Ans. £2043 0s. 11 $\frac{3}{4}$ d.

24. To Reduce Old Canadian Currency or any State Currency to dollars and cents:—

RULE.

Express the given sum decimally and divide it by the value of a dollar expressed as a fraction of a pound; the quotient will be dollars, cents, &c.

EXAMPLE 7.—Reduce £179 18s. 4 $\frac{1}{2}$ d., Old Canadian Currency, to dollars and cents.

OPERATION.

$$£179 \text{ 18s. } 4\frac{1}{2}\text{d.} = £179\cdot9197916 \text{ and } 179\cdot9197916 \div \frac{1}{4} = \$719\cdot67916. \textit{Ans.}$$

NOTE.—Old Canadian Currency may be most expeditiously reduced to dollars and cents by the rule given in Art. 80, Sect. I.

EXAMPLE 8. Reduce £234 18s. 9 $\frac{1}{2}$ d., Ohio Currency, to dollars and cents.

OPERATION.

$$£234 \text{ 18s. } 9\frac{1}{2}\text{d.} = £234\cdot9385416 \text{ and } 234\cdot9385416 \div \frac{2}{5} = \$587\cdot3465416. \textit{Ans.}$$

EXERCISE.

9. Reduce £743 18s. 11d., New England Currency, to dollars and cents. *Ans.* \$2479·8194.
10. Reduce £119 9s. 8 $\frac{1}{2}$ d., Maryland Currency, to dollars and cents. *Ans.* \$318·625.
11. Reduce £473 17s. 1 $\frac{1}{2}$ d., Georgia Currency, to dollars and cents. *Ans.* \$2030·816964.

25. To reduce dollars and cents to sterling money :—

RULE.

Divide the given sum by the value of £1 sterling (\$4·867), the quotient will be pounds sterling and decimal of a pound.

Reduce the decimal part (Art. 58, Sect. IV) to shillings & pence.

EXAMPLE 12.—Reduce \$749·83 to sterling money.

OPERATION.

$$74983 \div 4867 = £154·0641 = £154 \text{ 1s. } 3\frac{1}{2}\text{d. } \textit{Ans.}$$

EXERCISE.

13. Reduce \$1006·90 to sterling money. *Ans. £206 17s. 7½d.*
 14. Reduce \$916·87 to sterling money. *Ans. £188 7s. 8½d.*
 15. Reduce \$2114·81 to sterling money. *Ans. £434 10s. 4½d.*

26. To reduce sterling money to dollars and cents :—

RULE.

Express the given sum decimally and multiply by the legal value of £1 sterling (\$4·867).

EXAMPLE 16.—Reduce £78 11s. 4½d. to dollars and cents.

OPERATION.

$$£78 \text{ 11s. } 4\frac{1}{2}\text{d.} = £78·5697916, \text{ and } 78·5697916 \times 4·867 = \$382·309. \textit{Ans.}$$

EXERCISE.

17. Reduce £2043 11s. 3d. sterling to dollars and cents.
Ans. \$9946·01868.
 18. Reduce £777 7s. 7d. sterling to dollars and cents.
Ans. \$3783·50437.
 19. Reduce £557 19s. 5½d. sterling to dollars and cents.
Ans. \$2715·65418.

EXCHANGE.

27. Exchange is a commercial term, denoting the payment of money by a person residing in one place to a person residing in another, by draft or bill of exchange.

28. A Bill of Exchange is a written order addressed to a person directing him to pay, at a specified time and place, a certain sum of money to another person or his order.

29. The person who signs the bill of exchange is called *the drawer* or *maker* of the bill.

30. The person on whom it is drawn is called the *drawee*, and, after he has accepted it, the *acceptor*.

31. The person to whom the money is directed to be paid is called the *payee*.

32. The person who purchases the bill of exchange, i. e., the person in whose favor it is drawn, is called the *buyer* or *remitter*.

33. The person who has legal possession of the bill is called the *holder*.

34. The *acceptance* of a bill or draft is a promise on the part of the drawee to pay it at maturity or the specified time. The usual mode of accepting a bill is for the drawee to attach his signature to the word "*accepted*," written either across the face of the note or on its back.

NOTE.—A draft or bill of exchange should be presented to the drawee, for his acceptance, immediately on its receipt.

35. If the payee or holder of a bill or draft wishes to sell it or transfer it, he endorses it; i. e., he writes his name on the back.

NOTE.—If the endorser directs the bill to be paid to a particular person, the endorsement is called a *special endorsement*, and the person therein named is called the *endorsee*.

If the endorser simply writes his name on the back of the bill, the endorsement is called a *blank endorsement*.

When the endorsement is blank, or when the bill is made payable to bearer, it may be transferred from one to another at pleasure, and the drawee is bound to pay it to the holder at maturity. If the drawee or acceptor of a bill fail to pay it, the endorsers are responsible for the payment.

36. When the drawee of a bill refuses acceptance, or, having accepted, fails to make payment when it becomes due, the bill is immediately *protested*.

37. A *protest* is a formal declaration in writing, made by a public officer called a *Notary Public*, at the request of the holders of the bill, notifying the drawer, endorsers, &c., of its non-acceptance or non-payment.

NOTE.—If the drawer and endorsers are not notified within a reasonable time of the non-acceptance or non-payment of the bill, they are not responsible for its payment.

When a bill is protested for non-acceptance, the drawer must pay it immediately, even though the specified time has not arrived.

38. The time specified for the payment of a bill varies, and is a matter of agreement between the drawer and buyer. Some are payable at sight, some at a certain number of *days* or *months* after sight or after date. In both cases it is customary to allow *three days of grace*.

39. Bills of Exchange are divided into *inland* and *foreign bills*. When both *drawer* and *drawee* reside in the same country, they are called *inland bills* or *drafts*; when in different countries, *foreign bills*.

NOTE.—Three bills are commonly drawn for the same amount, &c., and are called respectively the *First*, *Second*, and *Third of Exchange*, and together constitute a set. These are sent by different ships or conveyances; and when the *first* that arrives is accepted or paid, the others become void. This plan is adopted in order to avoid the delays which might arise from accidents, miscarriage, &c.

FORM OF AN INLAND BILL OR DRAFT.

\$3000.

TORONTO, 1st July, 1859.

Ten days after sight, pay to the order of George McCallum, Esq., Three Thousand Dollars, value received, and charge the same to

RIDOUT & STEVEN.

Messrs. Hardman & Morris,
Bankers, Hamilton.

FORM OF A FOREIGN BILL OF EXCHANGE.

Exchange 8000 francs.

TORONTO, 17th July, 1859.

At sixty days sight of this first of exchange (the second and third of the same date and tenor unpaid) pay to Edward Atkinson, Esq., or order, the sum of Eight Thousand Francs, with or without further advice.

JOHN HENDERSON.

Messrs. Duhamel & Beaudrhuois,
Bankers, Paris.

40. The *par of exchange* is that amount of the money of one country actually equal to a given sum of the money of another, and is either *intrinsic* or *commercial*.

41. The *intrinsic par of exchange* is the *real value* of the money of different countries, as determined by the weight and purity of their standard coins.

Thus, the English sovereign is *intrinsically* worth \$7861 of the gold coin of the United States.

42. The *commercial par of exchange* is a comparison of the coins of different countries, according to their nominal or market value.

Thus, the English sovereign varies in market value from \$483 to \$485.

NOTE.—The *intrinsic par* is always the same as long as the standard coins are of the same kind, quantity, and quality of metal; the *commercial par* is determined by commercial usage, and fluctuates, being different at different times.

43. The *Course of Exchange* signifies the *current price* paid in one country for bills of exchange drawn on another.

NOTE.—The *course of exchange* is constantly fluctuating from various causes. When the exports of a country just equal its imports, the exchange will be at *par*; when the balance of trade is against a place, i. e. when its imports exceed its exports, bills on foreign countries will be *above par*, because there will be a greater demand for them to pay the bills due abroad; when the balance of trade is in favor of a country, i. e. when its exports exceed its imports, bills of exchange on foreign countries will be *below par* since fewer of them will be required.

The course of exchange can never very greatly exceed the *intrinsic par value*, because when the premium on bills of exchange becomes great it is less expensive to importers to pay for the insurance and transportation of bullion or coin to meet their payments than to transmit bills of exchange.

44. By an old act of Provincial Parliament it was enacted that £100 sterling or 100 sovereigns should be equivalent to £111½ Canadian money, i. e. to \$444½ or £1 sterling = \$444. It was found however that this was very much below the real or intrinsic value of the sterling pound, accordingly, while its legal value was only \$444, the market or commercial value varied from \$483 to \$486. By an act recently passed by the Provincial Parliament, the value of the pound sterling was fixed at \$486.

Now the new par is equal to the old par *plus* nine and a-half per cent. of the old par, that is, \$444 + 9½ per cent. of \$444, which is 422, make \$486 = the new par. Consequently the rate of exchange between Canada and Great Britain must reach the nominal premium of 9½ per cent. before it is at par, according to the new standard.

45. Rates of exchange between Canada and Great Britain are commonly reckoned at a certain per cent. on the old par of exchange, instead of on the new par.

EXAMPLE 1.—A merchant in Hamilton wishes to remit to London £749 3s. 6d. sterling; exchange being at 10 per cent. premium; how much must he pay for the bill of exchange?

OPERATION.

Old commercial par of £1 sterling = \$444

To which add 10 per cent. of itself = 444

Gives price of £1 = 4888

Then £749 3s. 6d. = £749.175 × 4888 = \$3662.63½. *Ans.*

EXAMPLE 2.—A merchant in Toronto wishes to remit 144479 francs to Paris, exchange being at a premium of 2 per cent. What will be the cost of his bill in dollars and cents?

OPERATION.

Commercial value of the franc = 186 cents.

Add 2 per cent. = 372 "

Gives value for remitting = 18972 "

Then 18972 × 144479 = \$2741055588. *Ans.*

EXAMPLE 3.—What sum in dollars and cents will purchase a bill of exchange on Hamburg for 14667 marcs banco, exchange being at 1½ per cent. discount?

OPERATION.

Commercial value of the marc banco = 35 cents.

Deduct 1½ per cent. = 525 "

Gives value for remitting = 34475 "

Then 34475 cents × 14667 = \$5056448. *Ans.*

EXERCISE.

4. If I wish to remit \$16785.25 to Paris, for how many francs and centimes can I obtain a bill—exchange being 5 francs 4 centimes to the dollar?
Ans. 84597 francs 66 centimes.
5. What is the cost of a bill of exchange for 4000 marcs banco at one per cent. above par?
Ans. \$1414.
6. How much must I give for a draft on New York for \$35678 at $2\frac{1}{4}$ per cent. premium?
Ans. \$36480.755.
7. What will a bill of exchange on St. Petersburg for 2560 rubles cost in dollars and cents, at 2 per cent. discount, the par being 75 cents per ruble?
Ans. \$1881.60.
8. What will be the cost of a bill of exchange on Great Britain for £800 sterling at 8 per cent. premium?
Ans. \$3840.00.

ARBITRATION OF EXCHANGE.

46. Arbitration of exchange is the process of changing a given amount of the money of one country into an equivalent sum of the money of another, through the medium of one or more intervening currencies with which the first and last are compared.

NOTE.—Arbitration enables a person to ascertain whether it is more advantageous to draw or remit a bill of exchange direct from one country to another or indirectly through other places.

47. When there is but *one* intervening country, the operation is termed *simple arbitration*; when there are *two* or *more* intervening countries, *compound arbitration*.

48. All questions in arbitration of exchange may be solved by one or more statements in simple proportion; it is more convenient, however, to consider them as problems in Conjoined Proportion, and work them by the rule given in Art. 50, Sec. V.

NOTE.—Care must be taken to reduce all the money of the same country to the same denomination before linking them as directed in the rule.

EXAMPLE 1.—A merchant in Toronto wishes to remit 2000 marcs banco to Hamburg, and the exchange between Toronto and Hamburg is 35 cents for one marc banco. He finds, however, that the exchange between Toronto and Lisbon is \$1.08 for 1 milree, that between Lisbon and Paris is 6 milrees for 38 francs, and that between Paris and Hamburg is 19 francs for 10 marcs banco. How much will he gain by the circuitous exchange?

OPERATION.

| STATEMENT. | | SAME CANCELLED. | |
|---|--|-----------------|--|
| 108 cents | = 1 milrec. | 108 = 1 | |
| 6 milrees | = 38 francs. | 6 = 38 | |
| 19 francs | = 10 marcs banco. | 19 = 10 | |
| 2000 marcs banco | = x. | 2000 = x | |
| | $x = 200 \times 3 \times 108 = \$648.$ | | |
| 2000 × 35 = \$700.00 = what he has to pay by direct exchange. | | | |
| 648.00 = what he has to pay by circuitous exchange. | | | |

Difference = \$ 52.00 = What he gains by the latter mode.

EXAMPLE 2—£824 Flemish being due to me at Amsterdam, it is remitted to France at 16d. Flemish per franc; from France to Venice at 300 francs per 60 ducats; from Venice to Hamburg at 100d. per ducat; from Hamburg to Lisbon at 50d. per 400 rees; from Lisbon to England at 5s. 8d. sterling per milree; and from England to Canada at \$4.867 per £1 sterling. Shall I gain or lose, and how much, the exchange between Canada and Amsterdam being 7s. 1d. Flemish per dollar?

OPERATION.

| STATEMENT. | | SAME CANCELLED. | |
|---------------|--|-----------------|-----------------|
| 16d. Flemish | = 1 franc. | 2 | 16 = 1 |
| 300 francs | = 60 ducats. | 50 | 300 = 60 |
| 1 ducat | = 100d. Flemish. | 1 | 100 = 1 |
| 50d. Flemish | = 400 rees. | 50 | 50 = 400 |
| 1000 rees | = 68d. British. | 10 | 1000 = 68 17 |
| 240d. British | = \$4.867. | 4 | 240 = 4.867 |
| x | = 197760d. Flemish. | x | 197760 = 197760 |
| | $\frac{17 \times 4.867 \times 3296}{2 \times 50} = \$2727.07\frac{1}{2}$ | | |

Then since exchange between Canada and Amsterdam is 7s. 1d. Flemish per dollar we have

$$\begin{aligned} 85d. \text{ Flemish} &= 100 \text{ cents.} \\ x \text{ " } &= 197760d. \text{ Flemish.} \end{aligned}$$

Here $x = \frac{197760 \times 100}{85} = \2326.47 = sum I should have received had it been transmitted direct from Amsterdam to Canada.

Hence by the circuitous exchange I gain the difference between \$2727.07½ and \$2326.47 that is \$400.60½.

EXERCISE.

3. If London would remit £1000 sterling to Spain, the direct exchange being 42½d. per piastre of 272 maravedis; it is asked whether it will be more profitable to remit directly, or to remit first to Holland at 35s. per pound; thence to France at 19½d. per franc; and thence to Spain at 360 maravedis per ducat? *Ans.* The circular exchange is more advantageous by 103 piastres, 3 reals, 20 maravedis.

4. A merchant wishes to remit \$4888.40 from Montreal to London, and the exchange is 10 per cent. He finds that he can remit to Paris at 5 francs 15 centimes to the dollar, and to Hamburg at 35 cents per marc banco. Now, the exchange between Paris and London is 25 francs 80 centimes for £1 sterling, and between Hamburg and London 13½ marcs banco for £1 sterling. How had he better remit?

Ans. If he remits direct to London he will obtain a bill for £1000.

If he remits through Paris he will obtain a bill for only £975 15s. 8½d.

If he remits through Hamburg he will obtain a bill for £1015 15s. 5d.

Hence the best way to remit is through Hamburg, and the next best way is direct to London.

5. A merchant in Quebec wishes to remit 1200 marcs banco to Hamburg, and the exchange of Quebec on Hamburg is 35 cents for 1 marc. He finds the exchange of Quebec on Paris is 18 cents for 1 franc; that of Paris on London, is 25 francs for £1 sterling; that of London on Lisbon, is 180 pence for 3 milrees; that of Lisbon on Hamburg, is 5 milrees for 18 marcs banco. How much will be gain by the circuitous exchange?

Ans. Direct exchange \$420; circuitous exchange \$375; gain \$45.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers after the questions refer to the numbered articles of the section.

1. What is profit and loss? (1)
2. How do we find the total gain or loss on a quantity of goods when the cost price and selling price are given? (2)
3. How do we find at what price an article must be sold so as to gain or lose a specified percentage, the cost price being given? (3)
4. How do we find the rate per cent. of profit or loss? (4)
5. How do we find the cost price when the selling price and the gain or loss per cent. are given? (5)
6. What is barter? (6)
7. What is alligation? (7)
8. Into what rules is alligation subdivided? (9)
9. What is alligation medial? (10)
10. What is alligation alternate? (11)
11. How is alligation alternate proved? (13)
12. Give the different rules for alligation. (12, 14-16)
13. What is meant by the exchange of currencies? (17)
14. What is meant by the currency of a country? (18)
15. How is the intrinsic value of a coin determined? (19)
16. What fixes the commercial value of a coin? (20)
17. How do you account for the fact that the \$ is of different values in the American States? (22)
18. Give the value of the pound currency in Canada, and in the different States.
19. How do we reduce dollars and cents to old Canadian currency or to any state currency? (23)

20. How do we reduce old Canadian currency or any state currency to dollars and cents? (24)
21. How do we reduce dollars and cents to sterling money? (25)
22. How do we reduce sterling money to dollars and cents? (26)
23. What is a bill of Exchange? (23)
24. Explain the terms *drawer*, *drawee*, *acceptor*, *payee*, *holder*, *endorser* and *endorsee*. (29-35)
25. How is a bill accepted? (33)
26. What is the difference between a blank endorsement and a special endorsement? (35)
27. What is meant by *protesting* a bill? (36, 37)
28. Explain what is meant by the First, Second, and Third of Exchange. (39)
29. What is the par of Exchange? (40)
30. Explain the difference between the intrinsic par and the commercial par of Exchange. (41, 42)
31. What is the course of Exchange. (43)
32. Explain what is meant by saying the par of Exchange between Canada and Britain is $9\frac{1}{2}$ per cent. (44)
33. Upon what is the rate of Exchange between Canada and Britain reckoned? (45)
34. What is arbitration of Exchange? (46)
35. What is the difference between simple and compound arbitration? (47)
36. By what rule are questions in arbitration of Exchange worked? (48)

SECTION X.

INVOLUTION, EVOLUTION, LOGARITHMS, AND LOGARITHMIC ARITHMETIC.

1. A power of any number is the product obtained by multiplying that number by itself one or more times.

Thus $25 = 5 \times 5$ is a power of 5; $81 = 3 \times 3 \times 3 \times 3$ is a power of 3, &c.

2. The number which, being multiplied once or oftener by itself, produces the power, is called the *root* of that power.

Thus 5 is the root of 25, since $5 \times 5 = 25$; 3 is the root of 81, since $3 \times 3 \times 3 \times 3 = 81$.

3. The powers of a number are called the *first*, *second*, *third*, *fourth*, *fifth*, &c., according as the root is taken *once*, *twice*, *thrice*, *four times*, *five times*, &c., as factor.

Thus, 81 is called the fourth power of 3, because 3 is taken 4 times as factor, in order to produce 81.

4. The second power of a number is also called its *square*, because a square surface, the length of one of whose sides is expressed by a given number, will have its area expressed by the second power of that number. (See Art. 62, Sec. I.)

5. The third power of a number is also called its *cube*; because if the length of one side of a cube be expressed by a given number, the solid contents of the cube will be expressed by the third power of that number. (See Art. 64, Sec. I.)

6. The *index* or *exponent* of a power is a small figure written to the right, indicating how often the root has to be taken as factor in order to produce the given power.

| | |
|---|-----------------------------|
| Thus, $2^1 = 2$ | $= 2 =$ First power of 2. |
| $2^2 = 2 \times 2$ | $= 4 =$ Second power of 2. |
| $2^3 = 2 \times 2 \times 2$ | $= 8 =$ Third power of 2. |
| $2^4 = 2 \times 2 \times 2 \times 2$ | $= 16 =$ Fourth power of 2. |
| $2^5 = 2 \times 2 \times 2 \times 2 \times 2$ | $= 32 =$ Fifth power of 2. |

So also 8^7 means the seventh power of 8; i. e., a number produced by taking 8 seven times as factor, &c.

7. $(5+8)^2$ means that the sum of 5 and 8 is to be squared as one number and is a very different thing from 5^2+8^2 , which means the sum of the squares of 5 and 8.

Thus $(5+8)^2 = 13^2 = 169$, while $5^2+8^2 = 25+64 = 89$.

Therefore $(5+8)^2 = 25+80+64 = 1st\ part\ squared,\ plus\ twice\ product\ of\ 1st\ part\ by\ 2nd\ part,\ plus\ 2nd\ part\ squared.$

8. The process of finding a power of a given number by multiplying it into itself is called *Involution*.

9. To involve a number to any required power:—

RULE.

Take the given number as factor as many times as there are units in the index of the required power and find the continued product of these factors.

NOTE.—Fractions are involved by multiplying both numerators and denominators as above, and mixed numbers should be reduced to fractions before applying the rule.

EXAMPLE 1.—What is the fifth power of 7?

OPERATION,

Here the index of the required power is 5 and hence the given number 7 must be taken 5 times as factor.

$$7 \times 7 \times 7 \times 7 \times 7 = 16807 \text{ Ans.}$$

EXAMPLE 2.—What is the third power of $\frac{3}{4}$?

$$\text{Ans. } \left(\frac{3}{4}\right)^3 = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64} \text{ Ans.}$$

EXERCISE.

3. Find the fifth power of 3. Ans. 243.
4. Required the tenth power of 20. Ans. 1024000000000.
5. Required the sixth power of 1.05. Ans. 1.340095640625.
6. Find the seventh power of $\frac{3}{4}$. Ans. $\frac{3^7}{4^7} = \frac{2187}{16384}$.
7. Find the fifth power of $\frac{5}{9}$. Ans. $\frac{5^5}{9^5} = \frac{3125}{59049}$.
8. Required the third power of $11\frac{1}{2}$. Ans. $1\frac{1}{2}^3 = 1481\frac{1}{8}$.

10. Let it be required to find the product of 4^4 by 4^2 .

$$4^4 = 4 \times 4 \times 4 \times 4 \text{ and } 4^2 = 4 \times 4. \text{ Therefore } 4^4 \times 4^2 = (4 \times 4 \times 4 \times 4) \times (4 \times 4) \\ = 1 \times 4 \times 4 \times 1 \times 4 = 4^6 = 4^4 + 2.$$

Hence two or more powers of the same number are multiplied together by adding their indices or exponents.

$$\text{Thus, } 6^5 \times 6^2 \times 6^3 = 6^{5+2+3} = 6^{10} \\ 5 \times 5^2 \times 5^3 \times 5^7 = 5^{1+2+3+7} = 5^{13}, \text{ \&c., \&c.}$$

11. Let it be required to divide 3^5 by 3^2 .

$$3^5 = 3 \times 3 \times 3 \times 3 \times 3 \text{ and } 3^2 = 3 \times 3. \\ \text{Therefore } 3^5 \div 3^2 = \frac{3^5}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = 3 \times 3 \times 3 = 3^3 = 3^{5-2}.$$

Hence, to divide one power of a number by another power of the same number, we subtract the index of the divisor from the index of the dividend.

$$\text{Thus, } 7^5 \div 7^3 = 7^{5-3} = 7^2 \\ 3^{11} \div 3^4 = 3^{11-4} = 3^7, \text{ \&c., \&c.}$$

12. Let it be required to find the third power of 7^2 .

$$(7^2)^3 = 7^2 \times 7^2 \times 7^2 = 7 \times 7 \times 7 \times 7 \times 7 \times 7 = 7^6 = 7^{2 \times 3}$$

Hence to find any required power of a given power, we multiply the index of the given power by the index of the required power.

$$\text{Thus, } (2^4)^5 = 2^4 \times 5 = 2^{20}; (3^2)^7 = 3^2 \times 7 = 3^{14}, \text{ \&c., \&c.}$$

EXERCISE.

- 9. Multiply together 4^2 , 4^4 , 4^5 , and 4^7 . *Ans.* 4^{18} .
- 10. Divide 13^{11} by 13^2 . *Ans.* 13^9 .
- 11. Find the fifth power of 3^3 . *Ans.* 3^{15} .
- 12. Find the value of $\{(7^4 \times 7^3) \div (7^2 \times 7^2)\}^6$ *Ans.* 7^{18} .
- 13. Find the value of $\{(5^3 \times 5^4 \times 5^{11} \times 5^9) \div (5^3 \times 5^2 \times 5^7 \times 5^5)\}^3$. *Ans.* 5^{10} .

EVOLUTION.

13. Evolution is the process of finding any required root of a given power.

NOTE.—Evolution is the reverse of involution; the latter teaches how to find a power of a number by multiplying it into itself; the former, how to find the root of a power by resolving it into *equal factors*. It follows that powers and roots are correlative terms.—If one number is a power of another the latter is a root of the former.

14. A root of a number may be indicated by either of two methods.

1st. By using $\sqrt{\quad}$, called the radical sign (Lat. *radix*, a root).

2nd. By using a fractional index having unity for its numerator, and the number expressing the degree of the root for denominator.

Thus, The square root of 7 is expressed either by $\sqrt{7}$ or by $7^{\frac{1}{2}}$

The cube root of 6 is " " $\sqrt[3]{6}$ or by $6^{\frac{1}{3}}$

The seventh root of 2 is " " $\sqrt[7]{2}$ or by $2^{\frac{1}{7}}$

NOTE.—The figure placed in the radical sign, or as denominator of the fractional index denotes the root.

A fractional index with *numerator* greater than one is sometimes used. In such cases, the *denominator* denotes the *root*, and the numerator the power to be taken.

Thus, $2^{\frac{2}{3}}$ means either the cube root of the square of 2 or the square of the cube root of 2.

The radical sign $\sqrt{\quad}$ a corrupted form of the letter *r*, the initial letter of the Latin word *radix*, "a root."

EXERCISE.

1. Express the square root of 17 and the cube root of 11.

$$Ans. \sqrt{17} \text{ or } 17^{\frac{1}{2}} \text{ and } \sqrt[3]{11} \text{ or } 11^{\frac{1}{3}}.$$

2. Express the fifth root of 4.

$$Ans. \sqrt[5]{4} \text{ or } 4^{\frac{1}{5}}.$$

3. Express the fourth root of 5^3 .

$$Ans. \sqrt[4]{5^3} \text{ or } 5^{\frac{3}{4}}.$$

4. Express the sixth root of 7^4 .

$$Ans. \sqrt[6]{7^4} \text{ or } 7^{\frac{4}{6}} = 7^{\frac{2}{3}}.$$

5. Express the third power of the fifth root of 2. *Ans.* $(\frac{2}{5})^3$ or $2^{\frac{3}{5}}$.

6. Express the eleventh power of the tenth root of 161.

$$Ans. (\sqrt[10]{161})^{11} \text{ or } 161^{\frac{11}{10}}.$$

15. Let it be required to extract the fifth root of 3^{15} .

The fifth root of 3^{15} is expressed either by $\sqrt[5]{3^{15}}$, or by $3^{\frac{15}{5}}$.

Taking the latter mode, we have $3^{\frac{15}{5}} = 3^3 = 3^{15} \div 5$.

Hence, to extract any root of a given power of a number, we divide the index of the power by the index of the root.

Thus, The seventh root of 2^{14} is $2^{14} \div 7 = 2^2$

The fourth root of 2^{12} is $2^{12} \div 4 = 2^3$, &c., &c.

EXTRACTION OF THE SQUARE ROOT.

16. To extract the square root of a number, is to find a number which, being multiplied once by itself, will produce the given number.

RULE.

I. Point off the given number into periods of two figures each, beginning at the decimal point.

II. Find the highest square contained in the left-hand period and place its root to the right of the number, in the place occupied by the quotient in division.

III. Subtract the square of the digit, put in the root from the left-hand period, and to the remainder bring down the next period, to the right, for a new dividend.

IV. Double the part of the root already found for a TRIAL DIVISOR.

V. Find how many times the trial divisor is contained in the dividend, exclusive of the right-hand digit, and place the figure thus obtained both in the root and also to the right of the trial divisor.

VI. Multiply the divisor thus completed by the digit last put in the root; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.

VII. Again, double the part of the root already found for a new TRIAL DIVISOR; proceed as in V. and VI., and continue the process until all the periods are brought down.

NOTE.—If the given number is not a perfect square, its exact square root cannot be found; but by annexing periods of ciphers, we can obtain any required approximation to it.

EXAMPLE 1.—What is the square root of 22420225?

| | |
|------------|------------------------------|
| 22420225 | (4735, is the required root. |
| 16 | |
| 87)642 | |
| 609 | |
| 943)3302 | |
| 2829 | |
| 9465)47325 | |
| 47325 | |

EXPLANATION.—Here 22 is the left hand period, and the highest square in 22 is 16, of which the square root is 4. We place 4 in the root and subtract 16 from 22. This leaves a remainder 6, to which we bring down the next period, 42, and thus obtain 642 for the new dividend. Our next step is to find the *trial divisor*, which we obtain by doubling the part of the root already found. This gives us 8, (= 4 doubled) and we ask how

many times 8 will go into 64 (the dividend exclusive of the right hand digit). Bearing in mind that we are to put the digit thus obtained both in the root and in the divisor, and that the completed divisor will be over 80, we find that the required digit is 7, which we accordingly place both in the root and in the divisor. The complete divisor is 87, which multiplied by 7, gives 609, and this subtracted from 642, gives a remainder 33, to which we bring down the next period, 02, and thus get 3302 for the next dividend.

Again, doubling the part of the root already found, we obtain 94 (= 47 doubled) for a trial divisor, and as this will go into 330 (the dividend exclusive of the right hand digit) 3 times; we place 3 both in the root and in the divisor.

Multiplying the 943 thus obtained by 3; subtracting and bringing down the next period, we get 47325 for the next dividend. The next trial divisor is 946 (= 473 doubled) which will go into 4732 (the dividend exclusive of the right hand figure) 5 times; and we therefore place 5 both in the root and in the divisor. Multiplying and subtracting, we find no remainder. 473 is therefore the square root of 22420225.

PROOF.— $4735 \times 4735 = 22420225$.

EXPLANATION AND REASON.

17. We may consider every number as consisting of its *tens* plus its *units*; that is, if the tens be represented by the letter *a* and the units by the letter *b*.

$$\text{Number} = a \times b; \text{ and}$$

$$\text{Number squared} = (a \times b)^2 = a^2 + 2ab + b^2.$$

Hence, the square of a number is equal to the square of the tens, plus twice the product of the tens by the units, plus the square of the units.

$$\text{Thus, } 69 = 60 + 9$$

$$\text{And } (69)^2 = (60+9)^2 = (60)^2 + 2 \times 60 \times 9 + 9^2 = 3600 + 1080 + 81 = 4761.$$

18. Let it now be required to extract the square root of 4761.

I. It is evident that the square of a number consisting of a single digit can never contain more than two digits or less than one; conversely the square root of a number of one or two digits must be a number of one digit. Again the square of a number consisting of two digits can never contain more than four or less than three digits; conversely the square root of a number of three or four digits must be a number consisting of two digits. Similarly, the square of a number consisting of three digits can contain neither more than six nor less than five digits, and conversely, the square root of a number consisting of five or six digits, must be a number of three digits, &c.; that is, one digit in the root is equivalent to two digits in the square or conversely, two digits in the square are equivalent to one digit in the root.

Hence, if we divide the given number into periods of two figures each beginning at the decimal point, the number of periods will indicate the number of digits in the root.

II. Taking the number 4761, we divide it into periods, thus, 47⁶¹, and since there are two periods in the square there must be two digits in the root. We thus learn that 4761 is the square of a certain number of tens, plus a certain number of units. Now it is manifest that the square of the tens can only be found in the second period, 47, since tens squared can give no digit of a lower order than hundreds. Also, that no part of the square of the units can be found in the second period, 47, since any single unit squared can give no digit of a higher order than tens.

Therefore the square of the units is found only in the first or lowest period, the square of the tens only in the second period, the square of the hundreds only in the third period, &c.

OPERATION.

$$\sqrt{4761} \text{ (69 = square root.}$$

$$36 \text{ = highest square in 2nd period.}$$

$$6 \text{ tens} \times 2 = 12 \text{ tens} + 9 \text{ units} = 129) \quad 1161 = \text{remainder which contains, 1st, twice product of tens by units, 2nd, the square of the units.}$$

$$1161 = \text{twice } 6 \text{ tens} \times 9 + 9^2.$$

III. In extracting the square root of this number, we look first for the digit occupying the place of tens in the root. We know (II.) that the square of tens is contained in the second period, 47, and the highest square contained in 47 must be the square of the highest digit that can possibly stand in the place of tens in the root. But the highest square in 47 is 36, the square root of which is 6. Placing 36 under the 47, 6 in the root, we subtract and bring down the next period, 61, and thus get a total remainder of 1161. Now

(Art. 17) the whole number 4761 consists of the square of the tens, plus twice the product of the tens by the units, plus the square of the units; and, since we have subtracted from it 36, (or if the ciphers be annexed 3600) the square of the tens, the remainder, 1161, must contain twice the product of the tens by the units, plus the square of the units; that is, twice 6 tens \times by a certain number of units, plus the square of that number of units; and because we do not know as yet what the units' figure of the root is, we use twice the tens for a trial divisor.

IV. Since we are now seeking the units' digit of the root, and since tens multiplied by units can give no digit of a lower order than tens, the right hand digit of the dividend can form no part of twice the product of the tens by the units, and we have simply to ascertain how often 12 tens (= twice 6 tens) will go in 116 tens. Evidently 9 times.

V. Lastly, we place the digit thus found in the root, because it is a figure of the root, and in the divisor, because the dividend contains not only twice the product of the tens by the units, but also the square of the units. Now when we multiply the 9 by 9 we get the square of the units, and when we multiply the 12 tens by 9 units, we get twice the product of the tens of the root by the units.

EXAMPLE 3.—Extract the square root of 127449.

OPERATION.

$$\begin{array}{r}
 \overset{\cdot}{1}\overset{\cdot}{2}\overset{\cdot}{7}\overset{\cdot}{4}\overset{\cdot}{4}\overset{\cdot}{9} \text{ (357)} \\
 \underline{9} \\
 65)374 \\
 \underline{325} \\
 707)4949 \\
 \underline{4949} \\

 \end{array}$$

EXPLANATION AND REASON.—From the pointing off we learn that the given number is the square of a certain number of hundreds, plus a certain number of tens, plus a certain number of units.

I. We are first then to look for the digit in the place of hundreds, and since hundreds squared can give no digit of a lower order than *tens* of thousands or of a higher order than hundreds of thousands, we see that the square of the hundreds can be found only in the left hand period. The highest square contained in the left hand period is 9, the square root of which is the left hand digit of the entire root.

II. After subtracting, we bring down the next period *only*, because we are now looking for the digit in the place of tens in the root. And since *tens* squared can give no digit of a lower order than *hundreds*, the lowest period cannot enter into any part of the square of *tens*, much less can it enter into any part of twice the product of the hundreds by the tens, and therefore when looking for the tens of the root, we pay no attention to the right hand period of the square.

III. The remainder of the process is similar and the reason for the various steps the same as in examples 1 and 2.

19. To extract the square root of a decimal:—

RULE.

I. Annex one cipher, if necessary, in order that the number of decimal places may be even.

II. Point off into periods of two figures each, beginning at the decimal point, and extract the square root as in whole numbers

remembering that the number of decimal places in the root will be equal to the number of periods in the square.

EXERCISE.

4. Extract the square root of 195364. *Ans.* 442.
 5. Extract the square root of .0676. *Ans.* .26.
 6. Extract the square root of 984064. *Ans.* 992.
 7. Extract the square root of 5, true to five decimal places.
Ans. 2.23606.
 8. Extract the square root of .5, true to six decimal places.
Ans. .707106.
 9. Extract the square root of 60.487129. *Ans.* 7.777.
 10. Extract the square root of 79792266297612001.
Ans. 282475249.
 11. Extract the square root of 0.0000012321. *Ans.* 0.00111.

20. To extract the square root of a fraction :—

RULE.

I. Reduce mixed numbers to improper fractions, and compound and complex fractions to simple ones, and the resulting fraction to its lowest terms.

II. Extract the square root of both numerator and denominator separately, if they have exact roots ; but if they have not both exact roots, reduce the fraction to its corresponding decimal, by Art. 56, Sec. IV., and then extract the root as in Art. 19.

EXAMPLE 12.—Extract the square root of $2\frac{1}{4}$.

OPERATION.

$$\text{Ans. } 2\frac{1}{4} = \frac{9}{4} \text{ and } \sqrt{\frac{9}{4}} = \frac{\sqrt{9}}{\sqrt{4}} = \frac{3}{2} = 1\frac{1}{2}.$$

EXAMPLE 13.—Extract the square root of $3\frac{3}{4}$.

OPERATION.

$$3\frac{3}{4} = \frac{15}{4} = 3.42857142 \text{ and } \sqrt{3.42857142} = 1.8516.$$

EXERCISE.

14. Find the square root of .1. *Ans.* $\frac{1}{10}$.
 15. Find the square root of $1\frac{1}{4}$. *Ans.* $1\frac{1}{2}$.
 16. Find the square root of $5\frac{1}{4}$. *Ans.* $2\frac{1}{2}$.
 17. Find the square root of $8\frac{1}{4}$. *Ans.* $2\frac{1}{2}$.
 18. Find the square root of $13\frac{1}{4}$. *Ans.* $3\frac{1}{2}$.

21. Let it be required to extract the square root of 63513.423 septenary.

| OPERATION. | EXPLANATION.— |
|--|--|
| $\begin{array}{r} 63513\overline{)4230(236\overline{)155} + \\ \underline{4} \\ 43)235 \\ \underline{162} \\ 466)4313 \\ \underline{4161} \\ 5051)122\overline{)42} \\ \underline{50\ 51} \\ 505\overline{)25}41\overline{)6130} \\ \underline{34\ 3564} \\ 505\overline{)335}4\overline{)223300} \\ \underline{3\ 436344} \\ \hline \overline{)453623} \end{array}$ | <p>We point off into periods of two places each, as in the decimal or common scale. Then the highest square in 6, the first period, is 4, of which the square root is 2. Subtracting 4 from the 6 and bringing down the next period, 35, we get 235 for the dividend. Next doubling the 2 we obtain 4, and we find that this will go into 23, the dividend exclusive of the right hand figure, 3 times. Placing this 3 in both root and divisor, multiplying (bearing in mind that 7 is the common ratio of the system) and subtracting, we obtain a remainder of 43, to which we bring down the next period, 13, and thus get 4313 for the next dividend, &c.</p> |

EXAMPLE 19.—Extract the square root of 4731392 *undenary* true to two places to the right of the separating point.

| OPERATION. | |
|--|--|
| 4731392(2182 ⁰⁰ <i>Ans.</i> | $\begin{array}{r} \overline{)4731392} \\ \underline{4} \\ 41)73 \\ \underline{41} \\ 428)3213 \\ \underline{3079} \\ 4352)11592 \\ \underline{8674} \\ 4354\overline{)9}3999\overline{)00} \\ \underline{3594\ 74} \\ 4355\overline{)79}404\overline{)0700} \\ \underline{360\ 5744} \\ \hline \overline{)55\ 5667} \end{array}$ |

EXERCISE.

20. Extract the square root of 11333311 *septenary*. *Ans.* 2626.
21. Extract the square root of 33233344 *senary*. *Ans.* 4344.
22. Extract the square root of 4234'10123 *quinary*. *Ans.* 43'412.
23. Extract the square root of 88888'888 *nonary*. *Ans.* 288'88.
24. Extract the square root of 248664ct69 *duodenary*. *Ans.* 54373.

APPLICATION OF SQUARE ROOT.

22. A triangle is a figure having three sides, and consequently three angles. When one of the angles is a *right angle*, like the corner of a square, the triangle is called a *right angled triangle*.

23. In a right angled triangle the side *opposite the right angle* is called the *hypotenuse*, and the sides containing the right angle, are called the *base* and the *perpendicular*.

24. It is shown by elementary geometry that the square described on the hypotenuse of a right angled triangle is equal to the sum of the squares described on the other two sides.

Or if h be the hypotenuse, b the base, and p the perpendicular; then

$$h^2 = b^2 + p^2, \text{ and hence}$$

$$h = \sqrt{b^2 + p^2}$$

$$b = \sqrt{h^2 - p^2}$$

$$p = \sqrt{h^2 - b^2}$$

That is—to find the hypotenuse of a right angled triangle when the other sides are given we add the square of the base to the square of the perpendicular and extract the square root of the sum.

To find the length of the base we subtract the square of the perpendicular from the square of the hypotenuse and extract the square root of the remainder.

To find the length of the perpendicular we subtract the square of the base from the square of the hypotenuse and extract the square root of the remainder.

25. The following principles are also established by geometry:—

Circles are to each other as the squares of their diameters.

If the diameter of a circle be multiplied by 3.1416, the product is the circumference.

If the square of half the diameter of a circle be multiplied by 3.1416, the product is the area.

If the square root of half the square of the diameter of a circle be extracted, it is the side of an inscribed square.

If the area of a circle be divided by 3.1416, the quotient is the square of half the diameter.

EXAMPLE 25.—If the hypotenuse of a right angled triangle is 12 feet long and the base 10 feet, how long is the perpendicular?

$$\begin{array}{r} \text{OPERATION.} \\ 12^2 = 144 \\ 10^2 = 100 \end{array}$$

$$\text{difference} = \overline{44} \text{ and } \sqrt{44} = 6.63324. \text{ Ans.}$$

EXAMPLE 26.—If the foot of a ladder be placed 20 feet from the side of a house, how long must it be in order to reach to the top of the house, the latter being 46 feet high?

OPERATION.

$$46^2 = 2116$$

$$20^2 = 400$$

$$\text{sum} = \overline{2516} \text{ and } \sqrt{2516} = 50.15. \text{ Ans.}$$

EXERCISE.

27. Suppose a ladder 100 feet long be placed 60 feet from the foot of a tree; how far up the tree will the top of the ladder reach? *Ans.* 80 feet.
28. Two persons start from the same place, and go, the one due north 50 miles, the other due west 80 miles. How far apart are they? *Ans.* 94.34 miles, nearly.
29. How large a square stick of timber can be hewn from a round stick 24 inches in diameter? *Ans.* 16.97 in. to the side.
30. A man has a ladder 36 feet long, which, when put on the outside of a ditch 20 feet wide, exactly reaches the top of the wall. Required the height of the wall. *Ans.* 29.933.
31. A ladder 40 feet long is placed against a wall 14 feet high, and just reaches the top; it is then turned over and touches the top of another wall 26 feet high. Required the breadth of the street. *Ans.* 22.622 yds.
32. If the area of a circle be 1760 yards, how many feet must there be in the side of a square to contain that quantity? *Ans.* 125.857.
33. A certain general has an army of 141376 men. How many must he place in rank and file to form them into a square? *Ans.* 376.
34. What is the distance through the opposite corners of a square yard? *Ans.* 4.24264 feet.
35. The distance between the lower ends of two equal rafters, in the different sides of a roof, is 32 feet, and the height of the ridge above the foot of the rafters is 12 feet. What is the length of a rafter? *Ans.* 20 feet.
36. What is the distance measured through the centre of a cube from one corner to its opposite corner, the cube being 3 feet, or 1 yard, on a side? *Ans.* 5.196 feet.
37. If an iron wire $\frac{1}{6}$ inch in diameter will sustain a weight of 450 pounds, what weight might be sustained by a wire an inch in diameter? *Ans.* 45000 lbs.
38. What length of rope must be tied to a horse's neck, in order that he may feed over an acre? *Ans.* 7.136 + perches.
39. Four men, A, B, C, D, bought a grindstone, the diameter of which was 4 feet; they agreed that A should grind off his share first, and that each man should have it alternately until he had worn off his share; how much did each man grind off?

NOTE.—In this question we disregard the thickness of the grindstone. After the first has ground off his portion, there will remain $\frac{1}{4}$ of the stone

Then the whole stone : part remaining :: square of diameter of whole stone : square of diameter of part remaining. (Art. 25)

That is, $1 : \frac{3}{4} :: 4^2 : x^2$, and hence $x = 4 \times \sqrt{\frac{3}{4}} = 4 \times \sqrt{.75} = .866 \times 4 = 3.461 =$ diameter of stone after the first has ground off his portion.

Similarly, after the second has ground off his portion there will remain $\frac{1}{2}$ of the stone, and after the third has taken his portion, $\frac{1}{4}$ of the stone.

Hence $1 : \frac{1}{2} :: 4^2 : x^2$, whence $x = 4 \times \sqrt{\frac{1}{2}} = 2.828$ ft. = diameter after 2nd has taken his portion.

$1 : \frac{1}{4} :: 4^2 : x^2$, whence $x = 4 \times \sqrt{\frac{1}{4}} = 2$ ft. = diameter after 3rd has taken off his portion.

| | | | |
|-------------------|-------------------|------------|---------------|
| Hence A takes off | 4—3.461 = | .539 ft. = | 6.432 inches. |
| B “ | 3.461—2.828 = | .633 ft. = | 7.632 inches. |
| C “ | 2.828—2 = | .828 ft. = | 9.936 inches. |
| D “ | remaining 2 ft. = | | 24 inches. |

CUBE ROOT.

26. To extract the cube root of a number is to find a number which taken *three times* as factor will produce the given number :—

RULE.

I. *Point off the number into periods of three figures each beginning at the decimal point.*

II. *Find the highest cube contained in the left hand period and place its root to the right of the number, in the place occupied by the quotient in division.*

III. *Subtract the cube of the digit put in the root from the left hand period, and to the remainder bring down the next period to the right for a new dividend.*

IV. *Multiply the square of the part of the root already found by 300 for a TRIAL DIVISOR.*

V. *Find how many times the trial divisor is contained in the dividend and put the figure thus obtained in the root.*

VI. *Complete the TRIAL DIVISOR by adding to it :*

1st. *The part of the root previously found \times the last digit put in the root \times 30 and*

2nd. *The square of the last digit put in the root.*

VII. *Multiply the divisor thus completed by the digit last put in the root ; subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

VIII. *Again multiply the square of the part of the root already found by 300 for a new TRIAL DIVISOR, find what digit to place next in the root as in V, complete the divisor by making the two additions to the trial divisor described in VI, multiply, subtract and bring down as directed in VII, and continue the process until all the periods are brought down.*

EXAMPLE 1.—What is the cube root of 429172932007 ?

OPERATION.

| | | | |
|--|--------------|--|-----------|
| | 429172932007 | | 7543 Ans. |
| | 343 | | |
| 1st trial divisor = $7^2 \times 300 =$ | 14700 | 86172 = 1st dividend. | |
| 1st increment = $7 \times 5 \times 30 =$ | 1050 | | |
| 2nd " = $5^2 =$ | 25 | | |
| 1st complete divisor = | 15775 | 78875 = product of comp. div by 5. | |
| 2nd trial divisor = $75^2 \times 300 =$ | 1687500 | 7297932 = 2nd dividend. | |
| 1st increment = $75 \times 4 \times 30 =$ | 9000 | | |
| 2nd " = $4^2 =$ | 16 | | |
| 2nd complete divisor = | 1696516 | 6786064 = product of comp. div. by 4. | |
| 3d trial divisor = $754^2 \times 300 =$ | 170554800 | 511868007 = 3rd dividend. | |
| 1st increment = $754 \times 3 \times 30 =$ | 67860 | | |
| 2nd " = $3^2 =$ | 9 | | |
| 3rd complete divisor = | 170622669 | 511868007 = product of comp. div. by 3. | |

EXPLANATION.—After pointing off we find that the highest cube number contained in the left hand period is 343 of which the cube root is 7. We therefore place 7 in the root and subtract 343 from the first period. This gives us a remainder of 86 to which we bring down the next period 172, and thus obtain 86172 for a new dividend.

Next we take 7, the part of the root already found, square it and multiply the 49 thus obtained by 300, this gives the first trial divisor 14700 which we find will go into the dividend 86172 (making due allowance for the increase of the divisor) 5 times.

Next we complete the divisor by adding to it.

1st, $7 \times 5 \times 30 = 1050$, and 2nd, $5^2 = 25$ which gives us 15775 for a complete divisor. This we multiply by 5, the digit last put in the root, subtract the product 78875 for the 1st dividend, and to the remainder 7297 bring down the next period 932, &c., &c.

27. EXPLANATION AND REASON.—We have seen (Art. 17) that we may consider every number as consisting of its *tens* plus its *units*, or if $a = \text{tens}$ and $b = \text{units}$, then

Number = $a + b$; and

Number cubed = $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Hence the cube of a number is equal to the cube of the tens, plus three times the product of the tens squared multiplied by the units, plus three times the product of the tens multiplied by the square of the units, plus the cube of the units.

Thus $69 = (60 + 9)$, and

$$\begin{aligned} 69^3 &= (60 + 9)^3 = 60^3 + 3 \times 60^2 \times 9 + 3 \times 60 \times 9^2 + 9^3 \\ &= 216000 + 97200 + 14580 + 729 \\ &= 328509. \end{aligned}$$

28. Let it now be required to extract the cube root of 328509.

I. It is manifest that the cube of a single digit can never contain more than three digits or less than one digit, and hence the cube root of a number (i. e., perfect cube) of one, two or three digits must be a number of one digit. Again the cube of a number consisting of two digits can never contain more than six or less than four digits, and conversely the cube root of a perfect cube consisting of four, five or six digits must be a number of two digits. Similarly the cube root of a perfect cube consisting of seven, eight or nine digits must be a number of three digits, &c.

Hence, one digit in the root is equivalent to three digits in the cube, and conversely three digits in the cube are equivalent to one digit in the root, and therefore if we divide the given number into periods of three digits each, beginning at the decimal point, the number of periods will indicate the number of digits in the root.

II. The cube of the units can be found only in the period immediately to the left of the decimal point, since any unit cubed can give no digit of a higher order than hundreds. Also the cube of the tens can be found only in the second period to the left of the decimal point, since tens cubed can give no digit of a higher order than *hundreds of thousands*, or of a lower order than *thousands*. Similarly the cube of the hundreds can be found only in the third period to the left of the decimal point, &c.

Hence, counting from the decimal point towards the left, the cube of the units can be found only in the first period, the cube of the tens only in the second period, the cube of the hundreds only in the third period, &c.

III. Taking the number 328509 we divide it into periods, thus 328509, and since there are two periods in the cube there must be two digits in the root. We thus learn that 328509 is the cube of a certain number of tens plus a certain number of units. We first then look for the digit in the place of tens in the root. We know (II.) that the cube of the tens is contained in the second period, 328, and the highest cube contained in 328 must evidently be the cube of the highest digit that can occupy the place of tens in the root—which digit we are seeking. The highest cube

| | | |
|------------------------------------|--------------|--|
| | OPERATION. | |
| | 328509(69 | |
| | 216 | |
| $6^2 = 36 \times 300 = 10800$ | 112509 | |
| $6 \times 9 = 54 \times 30 = 1620$ | 1620 | |
| $9^2 = 81$ | 81 | |
| | 12501 112509 | |

contained in 328 is 216, of which the cube root is 6. We then subtract 216 from 328 and to the remainder bring down 509, the next period, which gives us 112509 for a new dividend.

IV. From the given number we have only subtracted 216 (or if the ciphers be affixed, 216000) the remainder, 112509, therefore consists (Art. 27) of three times the product of the square of the tens by the units, plus three times the product of the tens by the square of the units, plus the cube of the units; that is, 112509 consists of $(6 \text{ tens})^2 \times 3 \times \text{a certain number of units} + (6 \text{ tens}) \times 3 \times (\text{that number of units})^2 + (\text{that number of units})^3$; and because we do not know as yet what the units' figure is, we use $(6 \text{ tens})^2 \times 3$ for a trial divisor.

But $(6 \text{ tens})^2 \times 3 = (60)^2 \times 3 = (6 \times 10)^2 \times 3 = 6^2 \times 10^2 \times 3 = 6^2 \times 300$; or, in other words, any number of tens squared, multiplied by 3, is equal to that same number of units squared and multiplied by 300. Hence we obtain the constant multiplier, 300.

V. $6^2 = 36$, and this multiplied by 300 gives us 10800. In asking how often this is contained in 112509 we have to bear in mind that we must increase the trial divisor by the two additions indicated in the sixth section of the rule. Making allowance for these additions, we find the units' figure of the root to be 9.

VI. If we were to multiply the 10800 we have obtained as a trial divisor by 9, the units' figure of the root, we should only get three times the product of the square of the tens by the units; but we require also three times the product of the tens by the square of the units and lastly the cube of the units. Our complete divisor must therefore evidently consist of—

1st. Three times the square of tens.

2nd. Three times the tens multiplied by the units.

3rd. The square of the units; or representing the tens by a and the units by b , the divisor must = $3a^2 + 3ab + b^2$, and this multiplied by b , the digit in the units' place will give

$$(3a^2 + 3ab + b^2)b = 3a^2b + 3ab^2 + b^3 = \text{the dividend.}$$

Now $(6 \text{ tens}) \times 3 = (60) \times 3 = 6 \times 10 \times 3 = 6 \times 30$ i.e. the product of any number of *tens* multiplied by 3, is equal to the product of that same number of *units* multiplied by 30.

Hence we obtain the constant multiplier 30.

The additions we make then are $6 \times 30 \times 9 = 1620$, and $9^2 = 81$, and thus we obtain the complete divisor $12501 = (60)^2 \times 3 + 60 \times 3 \times 9 + 9^2$, and multiplying this by 9, we get

$\{(60)^2 \times 3 + 60 \times 3 \times 9 + 9^2\} 9 = 60^2 \times 3 \times 9 + 60 \times 3 \times 9^2 + 9^3 =$ three times the square of the tens multiplied by the units, plus three times the tens multiplied by the square of the units, plus the cube of the units.

NOTE.—When there are more than two periods, the reasons are analogous, since we never have to do with more than *tens* and *units* of the root at one time; i.e., when we are seeking the second digit of the root, we call the first digit tens and the second, units; when we are seeking the third digit of the root we consider the first two as so many tens, and the third as units, &c.

The reason for bringing down only one period at a time is similar to the reason for the same step in the extraction of the square root (for which see Art. 18, Example 3).

29. To extract the cube root of a decimal :—

RULE.

I. *Annex two ciphers, if necessary, in order to make the last period complete.*

II. *Point off into periods of three places each, beginning at the decimal point, and extract the cube root as in whole numbers, remembering that the number of decimal places in the root will be equal to the number of periods in the cube.*

EXERCISE.

- | | |
|---|--------------------|
| 2. What is the cube root of 62712728317 ? | <i>Ans.</i> 3973. |
| 3. Extract the cube root of 1953125. | <i>Ans.</i> 125. |
| 4. Extract the cube root of 1076890625. | <i>Ans.</i> 1025. |
| 5. What is the cube root of .697864103 ? | <i>Ans.</i> .887. |
| 6. What is the cube root of 102503.232 ? | <i>Ans.</i> 46.8. |
| 7. Find the cube root of 179597.069288. | <i>Ans.</i> 56.42. |
| 8. Find the cube root of 483.736625. | <i>Ans.</i> 7.85. |
| 9. Find the cube root of .636056. | <i>Ans.</i> .86. |

30. To extract the cube root of a mixed number or a vulgar fraction :—

RULE.

I. Reduce mixed numbers to improper fractions, and compound or complex fractions to simple ones, and the resulting fraction to its lowest terms.

II. Extract the cube root of both numerator and denominator separately, if they have exact roots; but if they have not both exact roots, reduce the fraction to its corresponding decimal by Art. 56, Sect. IV, and then extract the root as in Art. 30.

EXAMPLE 10.—What is the cube root of $3\frac{2}{3}$?

OPERATION.

$$\sqrt[3]{3\frac{2}{3}} = \sqrt[3]{\frac{11}{3}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2} = 1\frac{1}{2}. \text{ Ans.}$$

EXAMPLE 11.—Extract the cube root of $17\frac{1}{2}$.

OPERATION.

$$17\frac{1}{2} = 17.125, \text{ and } \sqrt[3]{17.125} = 2.577, \text{ nearly.}$$

EXERCISE.

12. Extract the cube root of $\frac{2}{9}$. Ans. $\cdot 4721$.
13. Extract the cube root of $\frac{3}{7}$. Ans. $\cdot 5609$.
14. Extract the cube root of $\frac{1}{2}$ of $2\frac{1}{2}$. Ans. $\cdot 941$.
15. Extract the cube root of $28\frac{1}{2}$. Ans. $3\cdot 063$.
16. Extract the cube root of $32\frac{8}{11}$. Ans. $3\cdot 198$.

31. In extracting the cube root of a number in any scale, other than the decimal, we proceed in the same manner, pointing off into periods of three figures each, finding a trial divisor and afterwards completing it as in the preceding examples.

NOTE.—In all scales having a radix higher than 3, the constant multipliers are 300 and 30; but as in the *binary* and *ternary* scale we cannot use a digit so high as 3, these multipliers become respectively 1100 and 110 for the *binary* scale, and 1000 and 100 for the *ternary* scale.

EXAMPLE 17.—Extract the cube root of 613412·132 *septenary*.

OPERATION.

| | |
|---|---------------------|
| 613412·132 (65'04 | |
| 426 | |
| 6 ² = 51 × 300 = 21300 | 154413 |
| 6 × 30 = 210 × 5 = 1560 | 5 ² = 34 |
| 23224 | 152456 |
| 65 ² = 6304 × 300 = 2521500 | 1623'132 |
| 650 ² = 630400 × 300 = 252150000 | 1623'132000 |
| 650 × 30 = 26100 × 4 = 143400 | |
| 4 ² = 22 | |
| 252323422 | 1402'630321 |
| | 220'201346 |

EXERCISE.

18. Express one million in the *senary* scale and then extract its cube root. *Ans.* 244.
 19. Extract the cube root of 6131271 *octenary*. *Ans.* 165·32.
 20. Extract the cube root of 10221012·102 *ternary*.
Ans. 112·012.
 21. Extract the cube root of *teteet* in the *duodenary* scale true to two places to the right of the separating point.
Ans. e7·12.
 22. Extract the cube root of 421030·4412 *quinary* true to two places to the right of the separating point. *Ans.* 44·004.

32. Since many teachers prefer Horner's method of extracting the cube root to the common method, we shall give it here. Upon closely examining it the student will find that the reasons for the several steps of the process are identical with those given in Arts. 27 and 28. The constant multipliers 300 and 30 are still used, but in a disguised form.

RULE.

I. *Point off as in the common method.*

II. *Find the greatest cube in the first period on the left hand; place its root on the right of the number for the first figure of the root, and also in col. I. on the left of the number. Then multiplying this figure into itself, set the product for the first term in col. II.; and multiplying this term by the same figure again, subtract this product from the period, and to the remainder bring down the next period for a dividend.*

III. *Adding the figure placed in the root to the first term in col. I., multiply the sum by the same figure, add the product to the first term in col. II., and to this sum annex two ciphers, for a divisor; also add the figure of the root to the second term of col. I.*

IV. *Find how many times the divisor is contained in the dividend, and place the result in the root, and also on the right of the third term of col. I. Next multiply the third term thus increased by the figure last placed in the root, and add the product to the divisor; then multiply this sum by the same figure, and subtract the product from the dividend. To the remainder bring down the next period for a new dividend.*

V. *Find a new divisor in the same manner that the last divisor was found, then divide, &c., as before; thus continue the operation till the root of all the periods is found.*

EXAMPLE 23.—What is the cube root of 78314·6, true to two decimal places.

OPERATION.

| Col. I. | Col. II. | 78314'600 (42'78 +. |
|--------------------|-------------------------------|---------------------|
| 1st term <u>4</u> | <u>16</u> × 4 = | <u>64</u> |
| 2nd " <u>8</u> | <u>4800</u> , 1st divisor) | <u>14314</u> |
| 3rd " <u>122</u> | <u>5044</u> × 2 = | <u>10088</u> |
| 4th " <u>124</u> | <u>529200</u> , 2d divisor) | <u>4226600</u> |
| 5th " <u>1267</u> | <u>538069</u> × 7 = | <u>3766483</u> |
| 6th " <u>1274</u> | <u>54698700</u> , 3d divisor) | <u>460117000</u> |
| 7th " <u>12818</u> | <u>54801244</u> × 8 = | <u>438409952</u> |

EXPLANATION.—The cube root of the greatest cube in 78 is 4 which is placed in the root and also in column I, then multiplying this 4 by itself gives us 16 which is the 1st term in column II, and again multiplying this 16 by 4 gives us 64, the number which we are to subtract from the first period 78.

Subtracting and bringing down the next period 314 we get 14314 for the next dividend.

Now adding 4, the figure placed in the root, to 4 the 1st term in col. I, gives us 8, the 2nd term in col. I, multiplying this 8 by the 4, i. e., the figure in the root, gives us 32 which we add to the 1st term of col. II, and affix two ciphers. We thus obtain 4800 the second term of col. II, which is our trial divisor.

We then find that 4800 goes 2 times in the dividend. This 2 we place in the root and also to the right of the sum of the 1st and 2nd terms of col. I. The 1st and 2nd terms of col. I, added together make 12 and the 2 of the root affixed makes 122, the third term of col. I. Then we multiply this 122 by 2, the last digit put in the root, this gives us 244 which we add to 4800, the 2nd term of col. II, and thus obtain 5044, the 3rd term. Lastly this third term multiplied by 2, gives us the number to subtract, &c.

NOTE.—For examples in this method work any of the preceding questions.

APPLICATION OF THE CUBE ROOT.

33. *Principles Assumed.*—I. *Spheres are to one another as the cubes of their diameters.*

II. *Cubes and all other regular solids are to one another as the cubes of their like dimensions.*

EXERCISE.

- If a cannon ball 3 inches in diameter weighs 8 lbs., what will be the weight of a ball of the same metal 4 inches in diameter? $3^3 : 4^3 :: 8 \text{ lbs.} : \text{Ans.} = 18\frac{2}{3} \text{ lbs.}$
- If a ball 3 inches in diameter weighs 4 lbs., what will be the weight of a ball that is 6 inches in diameter? *Ans.* 32 lbs.
- If a globe of gold one inch in diameter be worth \$120, what is the value of a globe $3\frac{1}{2}$ inches in diameter? *Ans.* \$5145.
- If the weight of a well proportioned man, 5 feet 10 inches in height be 180 pounds, what must have been the weight of Goliath of Gath, who was 10 feet $4\frac{3}{4}$ inches in height?

Ans. 1015·1 lbs.

28. A person has a cube of clay whose sides are 973 ft. long; he wishes to take out of the same 5 cubes whose sides are 45 feet, 62 feet, 30 feet, 80 feet, and 20 feet. He requires to know the length of the side of the cube that can be formed out of the remaining clay. *Ans.* 972·69 ft.
29. What is the side of a cube which will contain as much as a chest 8 feet 3 inches long, 3 feet wide, and 2 feet 7 inches deep? *Ans.* 47·9843 inches.
30. Four ladies purchased a ball of exceeding fine thread, 3 in. in diameter. What portion of the diameter must each wind off so as to share off the thread equally?
Ans. 1st lady must wind off ·27432 inches.
 2nd " " " ·34458 "
 3rd " " " ·49122 "
 4th " " " 1·88988 "

NOTE.—This question is solved by a method similar to that adopted in Example 39 of the Square Root.

EXTRACTION OF THE ROOTS OF HIGHER ORDERS.

34. When the index of the root is a power of 2 or 3, or a multiple of any power of 2 by any power of 3—

RULE.

Resolve the given index into its prime factors.

Extract the root denoted by one of these factors, then of this root, extract the root denoted by another factor, and so on till all the prime factors be used.

Thus, for the 4th root extract the square root of the square root.
 for the 6th root extract the cube root of the square root.
 for the 8th root extract the square root of the square root of the square root.
 for the 12th root extract the cube root of the square root of the square root.
 for the 16th root extract the square root four times.
 for the 18th root extract the cube root of the cube root of the square root, &c., &c.

EXERCISE.

1. What is the fourth root of 19987173376? *Ans.* 376.
2. What is the sixth root of 308915776? *Ans.* 26.
3. Extract the ninth root of 40353607. *Ans.* 7.
4. Extract the eighteenth root of 387420489. *Ans.* 3.
5. Extract the twenty-seventh root of 134217728. *Ans.* 2.

LOGARITHMS.

35. The Logarithm of a number is the index of the power to which it is necessary to raise a given root or base, in order to produce the given number.

36. The Base of a system of logarithms is the *fixed number* to which all the logarithms of that system belong as indices.

Thus $10^3 = 1000$; here 3 is called the logarithm of 1000, to the base 10. So also $2^5 = 32$; here 5 is called the logarithm of 32, to the base 2, &c., &c.

37. A System of Logarithms is a collection of the logarithms of a series of numbers corresponding to the same base.

Any number whatever may be taken as the base of the system; but it is obvious that some numbers are much more convenient than others.

38. Two system of logarithms have been constructed and tables calculated with great care. They are,—

1st. The Common System or Briggsian System, whose base is 10.

2nd. Napierian System, whose base is 2.71828.

The Napierian System was invented by Baron Napier, and the peculiar base, 2.71828, was adopted chiefly because the logarithms having that base are more simply expressed and more easily calculated than any other. It has hence been called the *Natural System* of Logarithms. These logarithms were also formerly called *Hyperbolic* logarithms, from certain relations found to exist between them and the asymptotic spaces of the hyperbola, and which were erroneously believed to be peculiar to them.

The Common System was shortly afterwards invented by Briggs and adopted by Baron Napier, and is the system now universally employed for the purposes of calculation.

39. The *Characteristic* of a logarithm is the part which stands to the left of the decimal point.

40. The *Mantissa* (*handful*) is that part of the logarithm which stands to the right of the decimal point.

41. Since 10 is the base of the common system of logarithms and at the same time the radix of our system of notation, we have—

| | | | | | | |
|--------|----|-------------|-------------|--------|----|----|
| 100000 | == | 10^5 ; | whence log. | 100000 | == | 5 |
| 10000 | == | 10^4 ; | whence log. | 10000 | == | 4 |
| 1000 | == | 10^3 ; | whence log. | 1000 | == | 3 |
| 100 | == | 10^2 ; | whence log. | 100 | == | 2 |
| 10 | == | 10^1 ; | whence log. | 10 | == | 1 |
| 1 | == | 10^0 ; | whence log. | 1 | == | 0 |
| .1 | == | 10^{-1} ; | whence log. | .1 | == | -1 |
| .01 | == | 10^{-2} ; | whence log. | .01 | == | -2 |
| .001 | == | 10^{-3} ; | whence log. | .001 | == | -3 |
| .0001 | == | 10^{-4} ; | whence log. | .0001 | == | -4 |

42. From this it appears that the logarithm of any number between 1 and 10 will be more than 0 and less than 1; i. e., will be a fraction or a decimal; so also the logarithm of any number between 10 and 100 will be greater than 1 and less than 2; i. e., will be 1 and a fraction, or a decimal; so also the logarithm of any number between 100 and 1000 will be 2 and a decimal, &c.

Hence, the characteristic of any number containing digits to the left of the decimal point is positive and numerically *one less* than the number of such digits.

Thus, the characteristic of 7842 is 3; of 97826 it is 2; of 813426789 it is 8; of 390429 it is 0; of 26789426789 it is 4, &c.

43. It also appears, from Art. 41, that the logarithm of every number between 1 and 1 will be less than 0 and greater than -1; that is, it will be equal to -1, plus some decimal; the logarithm of every number between 1 and .01 will be less than -1 and greater than -2; or, in other words, will be -2 plus some decimal; so also the logarithm of every number between .01 and .001 will be -3 plus some decimal, &c., &c.

Hence, the characteristic of the logarithm of a decimal is negative and numerically *one greater* than the number of 0s which come between the decimal point and the first significant figure.

Thus, the characteristic of the logarithm of .000001 is $\bar{6}$; the characteristic of the logarithm of .0000000002347 is $\bar{11}$; the characteristic of the logarithm of .000278926345 is $\bar{4}$, &c., &c.

NOTE.—The negative sign affects only the characteristic—the mantissa or decimal portion of a logarithm is always positive. To indicate this it is customary to write the negative sign over the characteristic, as in the above examples, and not before it.

EXERCISE.

What are the characteristics of the logarithms of the following numbers:

1. 723, 91264, 81234567, 91267896124567, 23912342.

Ans. 2, 3, 4, 5, and 1

2. .027, .002134, .000000698, .8126714, .0000000002134.

Ans. $\bar{2}$, $\bar{3}$, $\bar{7}$, $\bar{1}$, and $\bar{10}$.

3. 1-1111111, 111111-11, 1000000000, .000000002162, $\bar{7}$, 12-78.

Ans. 0, 5, 9, 9, 0, and 1.

44. Since (Art. 11), to divide one power of a number by another power of the same we subtract the index of the divisor from the index of the dividend, and since common logarithms are indices to the base 10, let us take the number 47280 and successively dividing it by 10, examine the results.

| Numbers. | Logarithms. |
|---------------|---------------------|
| 47280 | = 4.674677 |
| 4728 | = 3.674677 |
| 472.8 | = 2.674677 |
| 47.28 | = 1.674677 |
| 4.728 | = 0.674677 |
| .4728 | = $\bar{1}$.674677 |
| .04728 | = $\bar{2}$.674677 |
| .004728 | = $\bar{3}$.674677 |

Here we have simply performed the same operation by two different methods, 1st. dividing the *numbers* by 10, and 2nd, from the *logarithms* corresponding to the numbers, subtracting 1, the logarithm of 10.

From this illustration it is evident that,—

1st. The characteristic of the logarithm of a number is dependent wholly upon the position of the decimal point in that number, and is not at all affected by the sequence of the digits that compose that number; and

2nd. The Mantissa or decimal part of the logarithm of a number is dependent wholly upon the sequence of the digits that compose that number, and is not at all affected by the position of the decimal point.

NOTE.—It is only common logarithms (i. e., those having 10 for their base) that possess the important property of having the same mantissa for the same figures, whether integral or decimal, or both, and it was this property that induced Briggs to adopt that base in preference to the Napierian base, 2.71828 .

45. Since the characteristic of the logarithm of any number does not depend upon the value of the digits composing that number, and is so easily found by attention to the rules found in Arts. 42, 43, it is customary to omit it altogether in logarithmic tables, and merely give the mantissa.

The annexed tables contain the logarithms of all numbers from 1 to 10000 calculated to 6 decimal places. When greater accuracy is required, tables calculated to a greater number of places are used. By means of the proportional parts and difference given in the tables, the logarithm corresponding to all numbers whatever, may be found with sufficient accuracy for all practical purposes.

46. To find the logarithm of any number not greater than 100 :—

RULE.

Find on the first page of the table of logarithms, the given number in the column marked No., and directly opposite to it,—in the column marked log., will be found the logarithm.

EXAMPLE 1.—What is the logarithm of 47? *Ans.* 1.672098.

NOTE.—By saying that 1.672098 is the logarithm of 47, we simply mean that the base 10, raised to the power 1.672098, is equal to 47, or briefly $10^{1.672098} = 47$.

EXAMPLE 2.—What is the logarithm of 93? *Ans.* 1.968483.

47. To find the logarithm of any number consisting of not more than four digits :—

RULE.

Find, in the column marked N, the first three digits of the given number.

Then the mantissa will be found in the intersection of the horizontal line containing these three digits and the vertical column at the head of which stands the fourth digit.

To this mantissa attach the characteristic as found by the rules in Art. 42.

EXAMPLE 3.—What is the logarithm of 7983 ?

Looking in the column marked N, we find the first three digits 798, on page 393 in the fourth horizontal division, counting from the top of the page and in the last line but one of that division. Carrying the eye along this horizontal line till we come to the vertical column, at the head of which stands the remaining digit, 3, we obtain for the mantissa of the required logarithm $\cdot 902166$, to which we prefix the characteristic 3 (since there are four digits to the left of the decimal point in the given number), and thus obtain the required logarithm $3\cdot 902166$.

EXAMPLE 4.—What is the logarithm of $\cdot 0000001234$?

The first three digits, viz : 123, are found in the fourth line of the third horizontal division on page 382, and at the intersection of this line with the column headed 4, is found $\cdot 091315$. To this we attach the characteristic $\bar{7}$, (since there are six 0s, between the decimal point and the first significant figure) and thus obtain the required logarithm, $\bar{7}\cdot 091315$.

EXERCISE.

5. What are the logarithms of 5794, 57·94, 5794000, and $\cdot 0005795$?

Ans. $3\cdot 762978$, $1\cdot 762978$, $6\cdot 762978$, and $\bar{4}\cdot 762978$.

6. What are the logarithms of 1·169, 11690, and $\frac{1169}{1000000}$?

Ans. $0\cdot 067815$, $4\cdot 067815$, and $\bar{3}\cdot 067815$.

7. What are the logs. of $\cdot 734$, 7340000000, and $\cdot 00000000734$?

Ans. $\bar{1}\cdot 865696$, $9\cdot 865696$, and $\bar{9}\cdot 865696$.

8. What are the logarithms of 978·4, 9·784, 978400, and $\cdot 9784$?

Ans. $2\cdot 990516$, $0\cdot 990516$, $5\cdot 990516$, and $\bar{1}\cdot 990516$.

48. To find the logarithm of a number containing more than four digits:—

RULE.

FIRST METHOD.—Find the mantissa corresponding to the logarithm of the first four digits by the last rule. Subtract this mantissa from the next following mantissa in the tables. Multiply the difference thus obtained by the remaining digits of the given number, and cut off from the product as many digits as there were in the multiplier (but at the same time adding unity if the highest cut off be not less than 5).

Add the number thus obtained to the mantissa of the logarithm corresponding to the first four digits, and the result will be the mantissa of the given number.

Lastly, attach the characteristic to this mantissa.

EXAMPLE 9.—What is the logarithm of 53803·2 ?

OPERATION.

The mantissa of the logarithm of 5380 (the first four digits) is $\cdot 730782$, and the next following mantissa is $\cdot 730863$.

Then from $\cdot 730863$
Subtract $\cdot 730782$

Difference $\quad 81$; and 81×32 (remaining digits of given number)

= 2592, from which we cut off *two* digits, since we multiplied by a number of two digits, and since the highest digit cut off is not less than 5, we add unity to the part retained, which gives us 26.

Then mantissa of logarithm of first four digits '730782
 Add 26

Mantissa of logarithm of given number '730808

To which attach the characteristic 4 and required logarithm = 4730808.

NOTE.—Except at the beginning of the tables, where the mantissas increase rapidly in magnitude, the difference may be taken from the right hand column, (headed D) and opposite the first three digits of the given number, where the mean difference of the mantissas in that line will be found.

EXAMPLE 10.—What is the logarithm of 832·17242?

OPERATION.

Mantissa of logarithm of 8321 '920176
 Difference from column D = 52; and $52 \times 7242 = 376584$ from which
 we cut off four digits and add..... 38

To which we attach the characteristic 2 and required logarithm = 2·920214

49. The difference given in the column headed D in the tables, is that due to an increment of *one unit* in the fourth figure of natural number, thus
 Logarithm of 5738..... 3·758761
 Logarithm of 5739..... 3·758836

Difference of natural numbers = 1; difference of logarithms = 75

And since it is shown in common works on Algebra that, with small increments in the natural numbers the logarithms corresponding to them increase in arithmetical progression, in order to find the logarithm of any number between those given above, we consider that the increment of the logarithm to be added to 3·758761, bears the same proportion to 75 (the increment for 1), that the increment of the natural number does to 1.

For example.—Let it be required to find the logarithm of 5738·47.

Here the increment of the given number being '47, we form the proportion 1 : '47 : : 75 : '47 \times 75 = 55·25, the increment to be added to 3·758761, and this addition having been made, we get 3·758796 for the logarithm of 5738·47.

Similarly, if the increment of the natural number had been '047 or '0047, the corresponding increment of the log. would have been '3525 or '3525.

These illustrations sufficiently explain the reasons of the last rule.

50. Taking the same number as in the last article and dividing the difference 75 by 10, we obtain 7·5 the difference corresponding to an increase of *one unit* in the *fifth* place of the natural number; the double of this, or 15 for two units, the treble or 22·5 for the three units, and so on; and each of the numbers thus obtained will be the increment of the logarithm corresponding to an increase of that number of units in the *fifth* place of the natural number. The increments thus obtained, and corresponding to each of the nine digits, are inserted in the left hand column of the tables, headed P. P. (Proportional Parts.)

51. The numbers in the column headed P. P., as already explained, are the increments in the logarithm for an increase in the *fifth* place of the natural numbers. They express also the increments for the digits in the *sixth*, *seventh*, *eighth*, *ninth*, &c., places of the natural number, when they are divided by 10, 100, 1000, &c., as the case may be.

52. Hence, to find the logarithm of any number containing more than four digits:—

RULE.

SECOND METHOD.—Find the mantissa of the logarithm corresponding to the first four digits of the given number.

Find in the same horizontal division as that in which the mantissa is found, the proportional part in the column headed P. P., corresponding to the digit in the fifth place of the given number, and set it down beneath the part of the mantissa already found, so that their right hand digits may be in the same vertical line. Find the P. P. corresponding to the digit in the sixth place of the given number, and set it down so that its right hand figure may be one place to the right of the last. Find the P. P. corresponding to the digit in the seventh place of the given number and set it down one place to the right of the last, and so on till all the digits of the given number be used.

Add the part of the mantissa already found, and the P. Ps. as written, together, and reject from the result all but the first six digits to the left, adding one to the last retained, if the highest of the rejected digits be not less than 5—the result will be the mantissa of the logarithm of given number.

Lastly, attach the proper characteristic to this mantissa, and the result will be the required logarithm.

EXAMPLE 11.—What is the logarithm of 8372.468 ?

OPERATION.

| | | |
|-------------------------------|---|--------|
| Mantissa of logarithm of 8372 | = | 922829 |
| P. P. corresponding to 4 | = | 21 |
| P. P. " to 06 | = | 31 |
| P. P. " to 008 | = | 42 |

Sum = 922853 | 52

Therefore required mantissa = 922854 and required log. = 3.922854.

EXAMPLE 12.—What is the logarithm of 403567 ?

OPERATION.

| | | |
|---------------------------------|---|--------|
| Mantissa of logarithm of 403500 | = | 605844 |
| P. P. corresponding to 60 | = | 64 |
| P. P. " to 7 | = | 75 |

Sum = 605915

Therefore required logarithm is 5.605916.

EXERCISE.

FIND THE LOGARITHMS OF THE FOLLOWING NUMBERS BY THE FIRST METHOD—OBTAINING THE DIFFERENCES BY SUBTRACTION.

13. What are the logarithms corresponding to 8193217, 73.9245, and .843742? *Ans.* 6.913455, 1.868789, and $\bar{1}$.926210.

14. Find the logarithms corresponding to .000234564 and .001007013. *Ans.* 4.370261 and $\bar{3}$.003033.

USING THE TABULAR DIFFERENCES.

15. Find the logarithms corresponding to 52·376 and 129·476.
Ans. 1·719133 and 2·112189.

USING THE PROPORTIONAL PARTS.

16. Find the logarithms corresponding to ·000471398 and 9136712.
Ans. $\bar{4}$ ·673387 and 6·960790.
17. Find the logarithms corresponding to 4·23429 and 763·12987.
Ans. 0·626780 and 2·882598.

53. To find the logarithm of a vulgar fraction :—

RULE.

Subtract the logarithm of the denominator from the logarithm of the numerator.

54. To find the logarithm of a mixed number :—

RULE.

Either reduce the mixed number to a fraction and proceed as in Art. 53, or reduce the fractional part to a decimal, attach it to the whole number and proceed as in Arts. 48-52.

55. To find the natural number corresponding to any given logarithm :—

RULE.

FIRST METHOD.—*Find that logarithm in the table which is next lower than the given one and the four digits corresponding to it will be the first four digits of the required number.*

II. *Subtract this logarithm from the given logarithm, to the remainder annex one cipher and divide by the tabular difference corresponding to the four digits already obtained, the quotient will be the fifth digit.*

III. *To the remainder attach another cipher and again divide by the tabular difference, the quotient will be the sixth digit and thus proceed till a sufficient number of digits has been obtained.*

IV. *The characteristic of the logarithm shows where to place the decimal point*

NOTE.—The number cannot be carried with accuracy to more places than the logarithm has decimal places. (See Art. 56.)

EXAMPLE 18.—Find the number corresponding to the logarithm 4·923267.

OPERATION.

| | | |
|---|-------------------------|--------------------------|
| Given log. | ·923267 | |
| Next lower in tables. | ·923244 = log. of 8380. | |
| Difference = | 23 | Tabular difference = 52. |
| Then $23000 \div 52$ gives 442 for digits in 5th, 6th and 7th places. | | |

Hence the digits of the natural number are 8350442; and since the characteristic is 4, *i.e.* one less than the number of digits to the left of the decimal point, the required number is 8350442.

SECOND METHOD.—Find the first four digits of the required number and also the difference between the given logarithm and the next lower in the table as in the last rule.

II. Find in the same horizontal division of the table the highest P. P. that does not exceed this difference. Opposite to it in the column headed N. will be found the digit of the fifth place.

III. Subtract this P. P. from the difference, to the remainder annex one cipher and find the highest P. P. not exceeding the number thus formed. Opposite to it in column N. will be found the sixth digit.

IV. Continue this process by the addition of ciphers till the required number of digits be found.

EXAMPLE 19.—Find the natural number corresponding to the logarithm 3.553259.

OPERATION.

| | | |
|--------------------------------------|------------------|-------------------------------------|
| Given log. | 3.553259 | |
| Next lower in table | 3.553155 | = log. of 3574 |
| | Difference = 104 | |
| Highest P. P. not greater than 104 = | 98 | corresponds to 8 for fifth [place. |
| | 60 | |
| Highest P. P. not greater than 60 = | 49 | corresponds to 4 in sixth [place. |
| | 110 | |
| Highest P. P. not greater than 110 = | 110 | corresponds to 9 in seventh [place. |
| | 110 | |

Therefore digits of required number are 3574849; and since the characteristic is 3, there must be four digits to the left of the decimal point.

Hence required number is 3574.849.

EXERCISE.

BY FIRST METHOD.

20. Find the natural numbers corresponding to the logarithms
 4.137139, 0.718134 and $\bar{4}$.635421.
Ans. 13713.227, 5.225278 and .0004319376.
21. Of what numbers are 2.921686 and $\bar{1}$.922165 the logarithms?
Ans. 835 and .8359211.

BY SECOND METHOD.

22. Of what numbers are 5.407968, 7.408386 and $\bar{3}$.416369 the logarithms?
Ans. 255839.4, 25608588 and .0026083.
23. What are the natural numbers corresponding to the logarithms 4.877777 and 0.555555?
Ans. 75470.5168 and 3.5938.

56. In order to ascertain how many figures of these results may be relied upon as correct, let us take from the tables any logarithm, as $4^{\circ}235683$.

Now the real value of this logarithm if carried to a greater number of places might be anything between $4^{\circ}2356335$ and $4^{\circ}2356345$, and might therefore differ from the given logarithm by very nearly $^{\circ}0000005$, which is therefore the extreme limit of the error attached to tables of six places; i. e. any difference less than $^{\circ}0000005$ might occur without producing any change in the logarithm as given in the table.

Now it is demonstrated in works treating of the theory of logarithms that the difference between the logarithms of numbers, which differ only by unity, is less than the modulus of the system divided by the smaller number. The modulus of the common system of logarithms is $^{\circ}4342945$, and if we let n represent the smaller number, the difference between the logarithms of n and of $n + 1$ is less than $^{\circ}4342945 \div n$.

Now we have shown that the difference between the true logarithm and that given in the table to six places, may be nearly equal to $^{\circ}0000005$, which is therefore less than $^{\circ}4322945 \div n$, or n is less than $\frac{^{\circ}4342945}{^{\circ}0000005}$. But $\frac{^{\circ}4342945}{^{\circ}0000005} = 868589$. That is, unless the number whose logarithm is given be less than 868589 its value cannot be found accurately beyond the first *five* digits, but if it be less than 868589, then the first *six* figures found from the table will be correct.

If tables of seven or eight places are used, the result can be depended on to seven or eight places, if the number be less than 868589 or if the mantissa be less than $^{\circ}8378$; but if greater, then the result can be relied on only to one less number of figures than the decimals of the logarithm.

LOGARITHMIC ARITHMETIC.

57. The *Arithmetical Complement* of a logarithm is the remainder obtained by subtracting the logarithm from 10.

Thus the arithmetical complement of $2^{\circ}713426$ is $10 - 2^{\circ}713426 = 7^{\circ}285574$.

EXERCISE.

Find the arithmetical complements of $5^{\circ}631642$ and $0^{\circ}714000$.

Ans. $4^{\circ}368358$ and $9^{\circ}286000$.

2. Find the arithmetical complements of $3^{\circ}123456$ and $7^{\circ}213149$.

Ans. $12^{\circ}876544$ and $16^{\circ}786851$.

3. Find the arithmetical complements of $6^{\circ}124357$ and $2^{\circ}000837$.

Ans. $3^{\circ}875643$ and $11^{\circ}999163$.

58. To multiply two or more numbers together by means of logarithms:—

RULE.

I. *Add their logarithms and the sum will be the logarithm of their product.*

II. *Find the natural number corresponding to this logarithm.*

NOTE 1.—For reason see Art. 10.

NOTE 2.—The following exercises are all worked by the difference and not by the proportional parts;

EXAMPLE 4.—Multiply 5631 by 47.

$$\begin{array}{l} \text{Logarithm of } 5631 = 3.750586 \\ \text{“ “ } 47 = 1.672098 \end{array}$$

$$\begin{array}{r} 5.422684 \\ 5.422590 = \text{logarithm of } 264800 \end{array}$$

$$\begin{array}{r} 94 = \\ \hline 57 \end{array}$$

Ans. 264800

EXERCISE.

5. Multiply 61, 22, and 65 together. Ans. 87230.
 6. Multiply 52, 734, and 6 together. Ans. 229008.
 7. Multiply together 35.86, 2.1046, .8372 and .00294. Ans. .185761.
 8. Multiply .00008764 by .86359. Ans. .000075685.

59. To divide numbers by means of their logarithms:—

RULE.

I. Subtract the logarithm of the divisor from the logarithm of the dividend: the result will be the logarithm of the required quotient.

II. Find the natural number corresponding to this.

NOTE.—For reason see Art. 11.

EXAMPLE 9.—Divide 6732.7 by 478.

OPERATION.

$$\begin{array}{l} \text{Logarithm of } 6732.7 = 3.828189 \\ \text{Logarithm of } 478 = 2.679428 \end{array}$$

$$\begin{array}{r} \text{Difference} = 1.148761 \\ 1.148603 = \text{logarithm of } 14.0800 \end{array}$$

$$\begin{array}{r} 158 = \\ \hline 51 \end{array}$$

Ans. 14.0851

EXAMPLE 10.—Divide .036584 by .00078593.

OPERATION.

$$\begin{array}{l} \text{Logarithm of } .036584 = \bar{2}.563291 \\ \text{Logarithm of } .00078593 = \bar{4}.895384 \end{array}$$

$$\begin{array}{r} \text{Difference} = 1.667907 \\ 1.667826 = \text{logarithm of } 46.5400 \end{array}$$

$$\begin{array}{r} 81 = \\ \hline 87 \end{array}$$

Ans. 46.5487

60. Instead of subtracting the logarithm of the divisor, we may add its arithmetical complement—the result, with 10 subtracted from the characteristic, will be the logarithm of the quotient.

Thus, in the last example the arithmetical complement of 4'895384 is 13'104616, and this added to 2'563291 gives 11'667907, and subtracting 10 from this characteristic, gives us 1'667907, the same as obtained by the other method.

NOTE.—This method of using the arithmetical complement is very convenient when we have to divide one number by the product of several others.

EXERCISE.

11. Divide $\cdot 6734$ by $\cdot 0009278$. *Ans.* 725'8033.
 12. Divide $437\cdot 89$ by $62\cdot 735$. *Ans.* 6'98.
 13. Divide $93\cdot 217$ by $\cdot 0007132$. *Ans.* 130702'4.
 14. Divide 9835267 by the product of 23, 189 and $2\cdot 748$.
Ans. 823'339.

61. To raise a quantity to any power by means of logarithms:—

RULE.

I. *Multiply the logarithm of the given number by the index of the required power, the result will be the logarithm of the required power.*

II. *Find the natural number corresponding to this logarithm.*

NOTE.—For reason see Art. 12.

EXAMPLE 15.—Find the 10th power of 2.

OPERATION.

Logarithm of 2 = 0.301030.

$0\cdot 301030 \times 10 = 3\cdot 010300 = \text{logarithm of } 1024$. *Ans.*

EXAMPLE 16.—Find the 7th power of $2\cdot 71$.

OPERATION.

Logarithm of $2\cdot 71 = 0\cdot 432969$.

Then $0\cdot 432969 \times 7 = 3\cdot 030783 = \text{logarithm of } 1073\cdot 45$. *Ans.*

NOTE.—In order to obtain the correct result when the characteristic happens to be negative, it must be recollected that the mantissa is *always* positive.

EXERCISE.

17. What is the 5th power of 5? *Ans.* 3125.
 18. What is the 6th power of $1\cdot 073$? *Ans.* 1'5261.
 19. What is the 4th power of $\cdot 0279$? *Ans.* $\cdot 00000060529$.
 20. What is the 11th power of $1\cdot 111$? *Ans.* 3'1831.

62. To extract any root of a given number by means of logarithms:—

RULE.

I. *Find the logarithm of the given number and divide it by the index of the required root, the result will be the logarithm of the root.*

II. Find the natural number corresponding to this logarithm.

NOTE.—For reason see Art. 15.

EXAMPLE 21.—What is the cube root of 12345 ?

OPERATION.

Logarithm of 12345 = 4·091491.

Then $4\cdot091491 \div 3 = 1\cdot363830 =$ logarithm of 23·11159. *Ans.*

63. To extract any root when the characteristic of the logarithm of the given number is negative :—

RULE.

I. If the characteristic is exactly divisible by the divisor, divide in the ordinary way, but make the characteristic of the quotient negative.

II. If the negative characteristic is not exactly divisible add what will make it so, both to it and to the decimal part of the logarithm. Then proceed with the division.

EXAMPLE 22.—Extract the fourth root of ·0076542.

OPERATION.

Logarithm of ·0076542 = $\bar{3}$ ·883899.

Now since $\bar{3}$ is not exactly divisible by 4 we add -1 to the characteristic and $+1$ to the mantissa which gives us $\bar{4} + 1\cdot883899$ and this is evidently = $\bar{3}$ ·883899.

Then $\bar{4} + 1\cdot883899 \div 4 = \bar{1}\cdot4709747 =$ logarithm of ·295784. *Ans.*

EXERCISE.

- | | |
|---|-----------------------|
| 23. Extract the 7th root of 913426000. | <i>Ans.</i> 19·0588. |
| 24. Extract the 11th root of 1·61342. | <i>Ans.</i> 1·04444 |
| 25. Extract the 5th root of ·000007139. | <i>Ans.</i> ·0934817. |
| 26. Extract the 7th root of ·002147. | <i>Ans.</i> ·41575. |

64. When the logarithms of two or more prime numbers are given, the logarithm of any multiples of these factors by each other can be easily obtained by attention to the foregoing rules.

Thus if the logarithm of 2 and 3 be given :—

1st. We can obtain the logarithm of any power of 2 or 3 by Art. 61, and any root of 2 or 3 by Art. 62.

2nd. We know the logarithm of 10, to be 1 and hence we can obtain the logarithm of 5 since $10 \div 2 = 5$ and also of 3·3 since $10 \div 3 = 3\cdot\bar{3}$, hence we can also obtain the logarithm of any power or root of 5 or 3·3.

3rd. By Arts. 58, 59, we can obtain the logarithm of any power or root of 2, 3, 5 and 3·3 multiplied by any power or root of 2, 3, 5 or 3·3.

EXAMPLE 27.—Given the logarithm of 2 = 0·301030 and the logarithm of 3 = 0·477121. Find the logarithms of 500, 24, 54, 120, 75000, $16\frac{2}{3}$, $\frac{1}{2}$, and $13\cdot5$.

OPERATION.

Since $5 = 10 \div 2$ the logarithm of 5 = $\log. 10 - \log. 2 = 1 - 0.301030 = 0.698970$.

Then logarithm of 500 = 2.698970.

$$24 = 8 \times 3 = 2^3 \times 3. \therefore \log. 24 = (\log. 2) \times 3 + (\log. 3.)$$

| | | | |
|----------|----------|-------|----------|
| log. 2 = | 0.301030 | × 3 = | 0.903090 |
| log. 3 = | 0.477121 | + 3 = | 477121 |

$$\text{Sum} = 1.380211 = \log. 24.$$

$$54 = 27 \times 2 = 3^3 \times 2. \therefore \log. 54 = (\log. 3) \times 3 + (\log. 2.)$$

| | | | |
|----------|----------|-------|----------|
| log. 3 = | 0.477121 | × 3 = | 1.431363 |
| log. 2 = | 0.301030 | + 3 = | 0.301030 |

$$\text{Sum} = 1.732393 = \log. 54.$$

$$120 = 4 \times 3 \times 10 = 2^2 \times 3 \times 10. \therefore \log. 120 = (\log. 2) \times 2 + (\log. 3) + (\log. 10.)$$

| | | | |
|-----------|----------|-------|----------|
| log. 2 = | 0.301030 | × 2 = | 0.602060 |
| log. 3 = | 0.477121 | + 3 = | 1 |
| log. 10 = | 1 | + 3 = | 1 |

$$\text{Sum} = 2.079181 = \log. 120.$$

75000 = 25 × 3 × 1000 = 5² × 3 × 1000. ∴ log. 75000 = (log. 5) × 2 + (log. 3) + (log. 1000.)

| | | | |
|-------------|----------|-------|----------|
| log. 5 = | 0.698970 | × 2 = | 1.397940 |
| log. 3 = | 0.477121 | + 3 = | 0.477121 |
| log. 1000 = | 3 | + 3 = | 3 |

$$\text{Sum} = 4.875061 = \log. 75000.$$

$16\frac{2}{3} = 3\frac{1}{3} \times 5$. ∴ logarithm of $16\frac{2}{3} = (\log. 3\frac{1}{3}) + (\log. 5)$.

Since $10 \div 3 = 3\frac{1}{3}$, $\log. 3\frac{1}{3} = \log. 10 - \log. 3 = 1 - 0.477121 = 0.522879$
 logarithm 5 = 0.698970

$$\text{Sum} = 1.221849 = \log. 16\frac{2}{3}.$$

$\frac{1}{2} = 5$. ∴ by changing only the characteristic = $\overline{1}.698970 = \logarithm \frac{1}{2}$.

$13.5 = 5 \times 27 = 5 \times 3^3$. ∴ logarithm 13.5 = (log. 3) × 3 + (log. 5.)

| | | | |
|---------------|----------|-------|----------|
| logarithm 3 = | 0.477121 | × 3 = | 1.431363 |
| logarithm 5 = | 0.698970 | + 3 = | 1.698970 |

$$\text{Sum} = 1.130333 = \log. 13.5.$$

EXERCISE.

28. Given logarithm 2 = 0.301030 and log. 7 = 0.845098, find the

logarithms of 14000, 4.9, .00196, 1750, 1428.571428, .00000112 and 3.0625.

$$\text{Ans. Log. 14000} = 4.146128.$$

$$\text{Log. 4.9} = 0.690196.$$

$$\text{Log. .00196} = \overline{3}.292256.$$

$$\text{Log. 1750} = 3.243038.$$

$$\text{Log. 1428.571428} = 3.154902.$$

$$\text{Log. .00000112} = \overline{6}.049218.$$

$$\text{Log. 3.0625} = 0.486076.$$

EXAMPLE 29.—Given logarithm $\frac{1}{2} = \bar{1}\cdot698970$
 logarithm $3 = 0\cdot477121$
 logarithm $11 = 1\cdot041393$

Find the logarithms of $49\frac{1}{2}$, 363 , $4\cdot09$, $2\cdot4$, $392\cdot72$, $293333\frac{1}{3}$ and $19\cdot965$.

Ans. Logarithm of $49\frac{1}{2} = 1\cdot694605$.
 Logarithm of $363 = 2\cdot559907$.
 Logarithm of $4\cdot09 = 0\cdot611819$.
 Logarithm of $2\cdot4 = 0\cdot388181$.
 Logarithm of $392\cdot72 = 2\cdot594090$.
 Logarithm of $293333\frac{1}{3} = 5\cdot467362$.
 Logarithm of $19\cdot965 = 1\cdot300270$.

QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers after the questions refer to the numbered articles of the section.

1. What is the power of a number? (1)
2. What is a root of a number? (2)
3. Why is the second power of a number called its square? (4)
4. Why is the third power of a number called its cube? (5)
5. What is the index or exponent of a power? (6)
6. What is involution? (8)
7. How do we multiply two or more different powers of the same number together? (10)
8. How do we divide any power of a number by another power of the same number? (11)
9. How do we find any required power of a given power? (12)
10. What is evolution? (13)
11. By what methods do we indicate a root of a number? (14)
12. How do we extract any root of a given power of a number? (15)
13. What is meant by extracting the square root of a number? (16)
14. What is the first step in extracting the square root of a number? (16)
15. Why do we point off into periods of two figures each? (18-I)
16. What is the second step in the process of extracting the square root? (16)
17. How do we know that the square root of the highest square in the left hand period is the highest digit of the root? (18-II)
18. What is the third step in the process of extracting the square root? (16)
19. Why do we bring down only the next period to the right? (18-II in Ex. 3)
20. What is the fourth part of the process for extracting the square root? (16)
21. Why do we double the part of the root already found for a trial divisor? (18-III)
22. What is the next step in extracting the square root of a number? (16)
23. Why do we not include the right hand figure of the dividend when seeking how many times the trial divisor is contained in it? (18-IV)
24. Why do we place the digit thus found in both the divisor and the root? (18-V)
25. What are the other steps used in extracting the square root? (16)
26. How do we extract the square root of a decimal? (19)

27. How do we extract the square root of a fraction or mixed number? (20)
 28. What is a triangle? (22) What is a right-angled triangle? (23)
 29. How may any one side of a right-angled triangle be found when the other two are given? (24)
 30. What proportion exists between different circles? (25)
 31. How may the area of a circle be found when its diameter is known? (25)
 32. What is meant by extracting the cube root of a number? (26)
 33. Give the different steps of the process of extracting the cube root. (26)
 34. If a number consist of a certain number of tens, plus a certain number of units, of what does its cube consist? (27)
 35. Why do we divide off into periods of three figures each? (28, I.)
 36. How do we know that the cube root of the highest cube contained in the left hand period is the highest digit of the root? (28, II.)
 37. Whence do we obtain, in the cube root, the constant multipliers 300 and 30. Illustrate by an example. (28, V. and VI.)
 38. Why do we make the two additions, indicated in the rule, to the trial divisor? (28, VI.)
 39. How do we extract the cube root of a decimal? (29)
 40. How do we extract the cube root of a fraction or mixed number? (30)
 41. In extracting the cube root of a number in any other scale, what changes must we make in the rule? (31)
 42. Give the different steps of Horner's method of extracting the cube root. (32)
 43. What proportion exists between the magnitude of similar solids? (33)
 44. How do we extract the higher roots when the index is a power of 2 or 3 or a multiple of 2 by 3? (34)
 45. What is a logarithm? (35)
 46. What is the base of a system of logarithms? (36)
 47. What is a system of logarithms? (37)
 48. What systems of logarithms have been constructed and how do they differ from one another? (38)
 49. What is the characteristic of a logarithm? (39)
 50. What is the decimal part of the logarithm called? (40)
 51. How do we find the characteristic of a logarithm? (42 and 43)
 52. Why is the negative sign written *over* the characteristic of the logarithm of a decimal? (43, Note.)
 53. Show that the characteristic of the logarithm of a number depends only on the position of the decimal point in the number, and the mantissa only in the sequence of figures. (44)
 54. Explain clearly what is meant by the numbers in column D of the tables. (43)
 55. Explain how the proportional parts in column P. P. are obtained. (50)
 56. Explain how the numbers in the column headed P. P. become the increments to be added to the logarithms for an increase in the sixth, seventh, eighth, &c., place in the natural number. (51)
 57. How do we find the logarithm of a vulgar fraction? (53)
 58. Explain to how many figures we may rely upon the accuracy of the results obtained by logarithmic tables. (56)
 59. What is the arithmetical complement of a logarithm? (57)
 60. How do we multiply numbers by means of their logarithms? (58)
 61. How do we divide numbers by means of their logarithms? (59, 60)
 62. How do we involve and evolve quantities by means of logarithms? (61, 62, 63)
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SECTION XI.

PROGRESSION, POSITION, COMPOUND INTEREST,
AND ANNUITIES.

PROGRESSION.

1. Quantities are said to be in *Arithmetical Progression* when they increase or decrease by a *common difference*.

Thus, 2, 5, 8, 11, 14, &c., are in arithmetical progression, the common difference being 3;

12, 10, 8, 6, &c., are in arithmetical progression, the common difference being 2.

2. In every progression the first and last terms are called the *extremes*, and the intermediate terms the *means*.

ARITHMETICAL PROGRESSION.

3. In *arithmetical progression* there are five things to be considered :

1. *The first term.*
2. *The last term.*
3. *The common difference.*
4. *The number of terms.*
5. *The sum of the series.*

These quantities are so related to one another that any three of them being given the other two can be found, and hence there are 20 distinct cases arising from these combinations.

4. If we represent these five quantities by letters, thus :

- a = the first term.
 l = the last term.
 d = the common difference.
 n = the number of terms.
 s = the sum of the series.

We shall be able easily to deduce algebraic formulas which, being interpreted, become the common arithmetical rules for arithmetical progression.

5. The general expression for an arithmetical series then become

$$a + (a+d) + (a+2d) + (a+3d) + (a+4d) + (a+5d) + \dots$$

where the coefficient of d is always 1 less than the number of the term. Thus in the third term the coefficient of d is 2, which is 1 less than the number of the term; in the fifth term the coefficient of d is 4, which is 1 less than the number of the term, &c.

Hence $l = a + (n-1)d$; that is, the last term of an arithmetical series is equal to the first term added to the product of the common difference by one less than the number of terms.

6. Since the sum of the series is equal to the sum of all the terms taken in any order whatever, we have

$$s = a + |a+d| + |a+2d| + |a+3d| + \dots + |l-3d| + |l-2d| + |l-d| + l$$

$$\text{Also } s = l + |l-d| + |l-2d| + |l-3d| + \dots + |a+3d| + |a+2d| + |a+d| + a$$

Hence $2s = (a+l) + (a+l) + (a+l) + (a+l) + \dots$ to n terms.

But $(a+l) + (a+l) + \dots$ to n terms $= (a+l)n$.

Therefore $2s = (a+l)n$, and dividing these equals by 2, we have $s = (a+l)\frac{n}{2}$. That is, *the sum of the series is found by adding together the first and last terms and multiplying their sum by half the number of terms.*

NOTE.—In adding the corresponding terms of the foregoing series together the d 's cancel out, thus adding the second terms of the right hand members together we have $a+d+l-d$, where the d 's cancel, and the sum becomes $d+l$: so also in the third terms we have $a+2d+l-2d = a+l$, &c.

7. From the formula obtained in Art. 5, we find by transposing the terms

$$l = a + (n-1)d$$

$$a = l - (n-1)d$$

$$d = \frac{l-a}{n-1}$$

$$n = \frac{l-a}{d} + 1$$

and substituting these values of l , a , d , and n in the formula obtained in Art. 6, we find

$$s = \left\{ 2a + (n-1)d \right\} \frac{n}{2}$$

$$s = \left\{ 2l - (n-1)d \right\} \frac{n}{2}$$

$$s = \frac{(l-a)(l+a)}{2d} + \frac{l+a}{2}$$

We thus obtain the five fundamental formulas from which the other fifteen are derived by transposing the terms, &c. Thus

$$l = a + (n-1)d \text{ gives formulas for } l, a, n, d = 4$$

$$s = (a+l)\frac{n}{2} \quad \text{“} \quad \text{“} \quad s, a, l, n = 4$$

$$s = \left\{ 2a + (n-1)d \right\} \frac{n}{2} \quad \text{“} \quad s, a, n, d = 4$$

$$s = \left\{ 2l - (n-1)d \right\} \frac{n}{2} \quad \text{“} \quad s, l, n, d = 4$$

$$s = \frac{(l+a)(l-a)}{2d} + \frac{l+a}{2} \quad \text{“} \quad s, a, l, d = 4$$

Total 20

8. THE FOLLOWING TABLE GIVES THE 20 FORMULAS FOR ARITHMETICAL PROGRESSION WITH THEIR RELATIONS, &C.

| No. | Given | Required. | Formulas. | Whence derived |
|--------|-----------|-----------|--|----------------|
| I. | a, d, n | | $l = a + (n-1)d$ | fundamental. |
| II. | a, d, s | | $l = -\frac{1}{2}d + \sqrt{2ds + (a - \frac{1}{2}d)^2}$ | VIII. |
| III. | a, n, s | l | $l = \frac{2s}{n} - a$ | V. |
| IV. | d, n, s | | $l = \frac{s}{n} + \frac{(n-1)d}{2}$ | VII. |
| V. | a, l, n | | $s = (a + l) \frac{n}{2}$ | fundamental. |
| VI. | a, d, n | s | $s = \left\{ 2a + (n-1)d \right\} \frac{n}{2}$ | V. and I. |
| VII. | d, l, n | | $s = \left\{ 2l - (n-1)d \right\} \frac{n}{2}$ | V. and XVII. |
| VIII. | a, d, l | | $s = \frac{(l+a)(l-a)}{2d} + \frac{l+a}{2}$ | V. and XIII. |
| IX. | a, n, l | | $d = \frac{l-a}{n-a}$ | I. |
| X. | a, n, s | | $d = \frac{2s-2an}{n(n-1)}$ | VI. |
| XI. | a, l, s | d | $d = \frac{(l+a)(l-a)}{2s-l-a}$ | VIII. |
| XII. | l, n, s | | $d = \frac{2nl-2s}{n(n-1)}$ | VII. |
| XIII. | a, d, l | | $n = \frac{l-a}{d} + 1$ | I. |
| XIV. | a, d, s | n | $n = \frac{d-2a}{2d} + \sqrt{\frac{2s}{d} + \left(\frac{2a-d}{2d}\right)^2}$ | VI. |
| XV. | a, l, s | | $n = \frac{2s}{l+a}$ | V. |
| XVI. | d, l, s | | $n = \frac{2l+d}{2d} + \sqrt{\left(\frac{2l+d}{2d}\right)^2 - \frac{2s}{d}}$ | VII. |
| XVII. | d, n, l | | $a = l - (n-1)d$ | I. |
| XVIII. | d, n, s | | $a = \frac{s}{n} - \frac{(n-1)d}{2}$ | VI. |
| XIX. | l, n, s | a | $a = \frac{2s}{n} - l$ | V. |
| XX. | d, l, s | | $a = \frac{1}{2}d + \sqrt{(l - \frac{1}{2}d)^2 - 2ds}$ | VIII. |

9. The following examples will enable the student to understand clearly the interpretation and application of these formulas :

10. To find the last term of an arithmetical series when the first term, the common difference, and the number of terms are given :—

RULE.

$$l = a + (n-1)d. \quad (i.)$$

INTERPRETATION.—The last term of a series is found by adding the first term to the product of the common difference by 1 less than the number of terms.

EXAMPLE 1.—What is the tenth term of the arithmetical series 1, 3, 5, &c.

OPERATION.

Here we have given the first term 1, the common difference 2 and the number of terms 10 to find the tenth or last term.

Then $l = a + (n-1)d = 1 + (10-1) \times 2 = 1 + 9 \times 2 = 1 + 18 = 19$. *Ans.*

11. To find the common difference of an arithmetical series when the first term, the last term, and the number of terms are given :—

RULE.

$$d = \frac{l-a}{n-1} \quad (ix.)$$

INTERPRETATION.—To find the common difference of an arithmetical series,—Subtract the first term from the last term and divide the difference thus obtained by one less than the number of terms.

EXAMPLE 2.—The first term of an arithmetical series is 3, the 13th term 55; find the common difference.

OPERATION.

Here we have given the first term 3, the last term 55, and the number of terms 13, to find the common difference.

Then $d = \frac{l-a}{n-1} = \frac{55-3}{13-1} = \frac{52}{12} = 4\frac{1}{3}$ = *Ans.*

12. To find the sum of an arithmetical series when the first term, the last term, and the number of terms are given :—

RULE.

$$s = (a+l) \frac{n}{2}. \quad (v.)$$

INTERPRETATION.—Add the first and last terms together and multiply their sum by half the number of terms.

EXAMPLE 3.—Find the sum of an arithmetical series whose first term is 2, last term 50 and number of terms 17.

OPERATION.

Here we have given the first term 2, the last term 50 and the number of terms 17 to find s , the sum of the series.

$$\text{Then } s = (a + l) \frac{n}{2} = (2 + 50) \times \frac{17}{2} = 52 + \frac{17}{2} = 26 \times 17 = 442. \text{ Ans.}$$

13. To find the common difference when the last term, the number of terms, and the sum of the series are given :—

RULE.

$$d = \frac{2nl - 2s}{n(n-1)}. \quad (\text{XII.})$$

INTERPRETATION.—Take twice the product of the number of terms by the last term, and from it subtract twice the sum of the series. Divide the resulting difference by the product of the number of terms by 1 less than the number of terms and the quotient will be the common difference.

EXAMPLE 4.—In an arithmetical series the last term is 80, the number of terms 11 and the sum of the series 746, required the common difference.

OPERATION.

Here we have given l , n , and s to find d and since $l=80$, $n=11$ and $S=746$ we have :

$$d = \frac{2nl - 2s}{n(n-1)} = \frac{(2 \times 11 \times 80) - (2 \times 746)}{11 \times (11-1)} = \frac{1760 - 1492}{11 \times 10} = \frac{268}{110} = 2\frac{3}{5}. \text{ Ans.}$$

14. To find the number of terms of an arithmetical series when the first term, the common difference, and the sum of the series are given :—

RULE.

$$n = \frac{d-2a}{2d} + \sqrt{\frac{2s}{d} + \left(\frac{2a-d}{2d}\right)^2}$$

INTERPRETATION.—I. Subtract the common difference from twice the first term, divide the remainder by twice the common difference, square the quotient, add the result to the quotient obtained by dividing twice the sum of the series by the common difference and extract the square root of this sum.

II. Next, from the common difference subtract twice the first term, divide the remainder by twice the common difference, and to the quotient add the square root obtained in I. The sum will be the number of terms.

EXAMPLE 5.—The first term of an arithmetical progression is 7, the common difference $\frac{1}{2}$, and the sum of all the terms 142. What is the number of terms?

OPERATION.

Here we have given a , d , and S , to find n and since $a = 7$, $d = \frac{1}{2}$, and $s = 142$, we have

$$n = \frac{d-2a}{2d} + \sqrt{\frac{2s}{d} + \left(\frac{2a-d}{2d}\right)^2} = \frac{\frac{1}{2}-2 \times 7}{2 \times \frac{1}{2}} + \sqrt{\frac{142 \times 2}{\frac{1}{2}} + \left(\frac{2 \times 7 - \frac{1}{2}}{2 \times \frac{1}{2}}\right)^2} =$$

$$\frac{\frac{1}{2}-14}{\frac{1}{2}} + \sqrt{\frac{284}{\frac{1}{2}} + \left(\frac{14-\frac{1}{2}}{\frac{1}{2}}\right)^2} = -\frac{13\frac{1}{2}}{\frac{1}{2}} + \sqrt{1136 + \left(\frac{13\frac{1}{2}}{\frac{1}{2}}\right)^2} = -27\frac{1}{2} +$$

$$\sqrt{1136 + (27\frac{1}{2})^2} = -27\frac{1}{2} + \sqrt{1136 + 756\frac{1}{4}} = -27\frac{1}{2} + \sqrt{1892\frac{1}{4}} = -27\frac{1}{2} + 43\frac{1}{2}$$

= 16. *Ans.*

EXERCISE.

- In an arithmetical series the first term is 4, the number of terms 17 and the sum of the series 884. What is the last term? *Ans.* 100.
- The extremes of an arithmetical series are 21 and 497, and the number of terms is 41. What is the common difference? *Ans.* $11\frac{1}{5}$.
- In an arithmetical series, the first term is 12, the last term 96 and the common difference is 6. Required the number of terms? *Ans.* 15.
- In an arithmetical series the last term is 14, the common difference 1 and the sum of the series 105. Required the number of terms? *Ans.* 15.
- The first term of an arithmetical series is $\frac{2}{3}$, the common difference $\frac{1}{3}$ and the sum of the series 1180. What is the last term? *Ans.* $39\frac{1}{3}$.
- If the extremes of an arithmetical series are 8 and 170 and the sum of the series 4895, what is the common difference? *Ans.* 3.
- If the extremes of an arithmetical series are 5 and $27\frac{1}{2}$ and the common difference $2\frac{1}{4}$, what is the number of terms? *Ans.* 11.
- If the first term of a series is 2, the last term 478 and the number of terms 86, what is the sum of the series? *Ans.* 20640.
- In an arithmetical series the last term is 998, the first term 2 and the common difference 6. What is the sum of the series? *Ans.* 83500.
- In an arithmetical series the first term is 5, the number of terms 11 and the common difference $2\frac{1}{4}$. What is the last term? *Ans.* $27\frac{1}{4}$.
- In an arithmetical series the last term is 199, the common difference is 11 and the number of terms 19. Required the sum of the series? *Ans.* 1900.
- The sum of an arithmetical series is 39840, and the extremes are 2 and 478. What is the number of terms? *Ans.* 166.
- The sum of an arithmetical series is 83500 and the extremes are 998 and 2. Required the common difference? *Ans.* 6.

19. A snail crawls up a flag staff 130 feet high and upon reaching the top begins to descend. In what time will he again reach the ground if he goes 2 feet the first day, 4 feet the second, 6 feet the third, and so on.
Ans. 15 days, 15 hours, 10 min. 27-264 sec.
20. The sum of an arithmetical series is 83500, the first term is 2 and the common difference 6, what is the last term?
Ans. 998.
21. A person wishes to discharge a debt of \$1125 in 18 annual payments which shall increase in arithmetical progression. How much must his first payment be in order that the last may be \$120?
Ans. \$5.
22. In an arithmetical series the extremes are 5 and $27\frac{1}{2}$ and the number of terms is 11. What is the common difference?
Ans. $2\frac{1}{2}$.
23. 220 stones are placed in a straight line exactly $2\frac{1}{2}$ yards apart, the first being $2\frac{1}{2}$ yards from a basket, how far will a person go whilst picking up the stones, returning with one at a time and depositing it in the basket?
Ans. $69\frac{1}{15}$ miles.
24. The sum of an arithmetical series is 39840, the number of terms is 166 and the last term is 478. What is the first term?
Ans. 2.
25. A person travelled from Toronto to Kingston, in 12 days, walking 4 miles the first day, 6 miles the second, 8 miles the third, and so on. How far is Toronto from Kingston?
Ans. 180 miles.
26. The clocks of Venice strike from 1 to 24. How many strokes does one of these clocks make in the day?
Ans. 300.

GEOMETRICAL PROGRESSION.

15. Quantities are said to be in Geometrical Progression when they increase or decrease by a common multiplier.

Thus 3, 12, 48, 192, &c., are in geometrical progression, the common ratio or common multiplier being 4.

100, 20, 4, $\frac{2}{5}$, $\frac{2}{25}$, &c., are in geometrical progression, the common ratio being $\frac{1}{5}$.

16. In *geometrical progression* there are five things to be considered :

1. *The first term.*
2. *The last term.*
3. *The common ratio.*
4. *The number of terms.*
5. *The sum of the series.*

As in arithmetical progression, these five quantities are so related that any three of them being given the other two can be found, and hence there are 20 distinct cases arising from their combinations.

17. Representing these five quantities by letters, thus,

a = the first term.

l = the last term.

r = the common ratio.

n = the number of terms.

S = the sum of the series.

the general expression for a geometrical series becomes

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots, \&c.,$$

where the index of r is always one less than the number of the term.

Thus in the third term the index of r is 2, which is one less than the number of the term; in the fifth term the index of r is 4, which is one less than the number of the term, &c.

Hence $l = ar^{n-1}$; that is, the last term is equal to the first term multiplied by the common ratio raised to that power which is indicated by one less than the number of terms.

18. Since the sum of the series is equal to the sum of all the terms.

$S = a + ar + ar^2 + ar^3 + \dots + ar^{n-3} + ar^{n-2} + ar^{n-1}$, multiplying by r , we get

$$Sr = ar + ar^2 + ar^3 + \dots + ar^{n-2} + ar^{n-1} + ar^n$$

Hence $Sr - S = ar^n - a$; or $S(r-1) = a(r^n - 1)$, and therefore $S = \frac{a(r^n - 1)}{r - 1}$

That is, the sum of the series is found by finding that power of the common ratio which is expressed by the number of terms—subtracting 1 from this, dividing the remainder by one less than the common ratio and multiplying the quotient by the first term.

NOTE.—The second of the above series is found from the first by multiplying both sides of the equation by r , and in subtracting we take the terms of the upper series from the corresponding terms of the lower. Only the first three or four and the last three or four terms are written and between ar^3 and ar^{n-3} there may be any number of intermediate terms. The ar^{n-3} in the lower series is obtained by multiplying the term before ar^{n-3} in the upper series, which is ar^{n-4} , by r .

19. From the formula obtained in Art. 17 we get by transposing the terms, &c.

$$l = ar^{n-1}$$

$$a = \frac{l}{r^{n-1}}$$

$$r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$$

$$= \frac{\log. l - \log. a.}{\log. r} + 1.$$

And substituting these values of l , a , r , n in the formula obtained in Art. 18 we find

$$s = \frac{rl-a}{r-1}$$

$$s = \frac{l(r^n-1)}{(r-1)r^{n-1}}$$

$$s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$$

and these together with the two formulas obtained in Arts. 17 and 18,

$$s = \frac{a(r^n-1)}{r-1}$$

$$l = ar^{n-1}$$

are the fundamental formulas of geometrical progression from which the other fifteen are derived by reduction. Thus,

$$s = \frac{rl-a}{r-1} \text{ gives formulas for } s, r, l, \text{ and } a = 4$$

$$s = \frac{l(r^n-1)}{(r-1)r^{n-1}} \quad \text{“ “ } s, r, l, \text{ and } n = 4$$

$$s = \frac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}} \quad \text{“ } s, l, n, \text{ and } a = 4$$

$$s = \frac{a(r^n-1)}{r-1} \quad \text{“ “ } s, r, a, \text{ and } n = 4$$

$$l = ar^{n-1} \quad \text{“ “ } l, a, r, \text{ and } n = 4$$

Total 20

20. The following table gives the 20 formulas for geometrical progression, with their relations, &c. It will be observed that questions involving formulas III, XII, XIV, and XVI cannot be solved by common arithmetic, but require the aid of the higher mathematics. All the formulas for n involve the use of logarithms.

| No. | Given. | Required. | Formulas. | Whence derived. |
|--------|------------|-----------|---|-----------------|
| I. | $a, r, n,$ | | $l = ar^{n-1}$ | fundamental. |
| II. | $a, r, s,$ | l | $l = \frac{a+(r-1)s}{r}$ | VI. |
| III. | $a, n, s,$ | | $l(s-l)^{n-1} - a(s-a)^{n-1} = 0$ | VII. |
| IV. | $r, n, s,$ | | $l = \frac{(r-1)sr^{n-1}}{r^n - 1}$ | VIII. |
| V. | $a, r, n,$ | | $s = \frac{a(r^n - 1)}{r^n - 1}$ | fundamental. |
| VI. | $a, r, l,$ | | $s = \frac{rl - a}{r - 1}$ | V. and I. |
| VII. | $a, n, l,$ | s | $s = l \frac{\frac{n}{n-1} - \frac{n}{n-1}}{\frac{1}{l^{n-1}} - \frac{1}{a^{n-1}}}$ | V. and XIII. |
| VIII. | $r, n, l,$ | | $s = \frac{l(r^n - 1)}{(r - 1)r^{n-1}}$ | V. and IX. |
| IX. | $r, n, l,$ | | $a = \frac{l}{r^{n-1}}$ | I. |
| X. | $r, n, s,$ | a | $a = \frac{(r-1)s}{r^n - 1}$ | V. |
| XI. | $r, l, s,$ | | $a = r(l-1)s$ | VI. |
| XII. | $n, l, s,$ | | $a(s-a)^{n-1} - l(s-l)^{n-1} = 0$ | VII. |
| XIII. | $a, n, l,$ | | $r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}$ | I. |
| XIV. | $a, n, s,$ | r | $r^n - \frac{s}{a}r + \frac{s-a}{a} = 0$ | V. |
| XV. | $a, l, s,$ | | $r = \frac{s-a}{s-l}$ | VII. |
| XVI. | $n, l, s,$ | | $r^n - \frac{s}{s-l}r^{n-1} + \frac{l}{s-l} = 0$ | VIII. |
| XVII. | $a, r, l,$ | | $n = \frac{\log. l - \log. a}{\log. r} + 1$ | I. |
| XVIII. | $a, r, s,$ | | $n = \frac{\log. [a + (r-1)s] - \log. a}{\log. r}$ | V. |
| XIX. | $a, l, s,$ | n | $n = \frac{\log. l - \log. a}{\log. (s-a) - \log. (s-l)} + 1$ | VII. |
| XX. | $r, l, s,$ | | $n = \frac{\log. l - \log. [rl - (r-1)s]}{\log. r} + 1$ | VIII. |

APPLICATIONS.

21. Given the first term, the common ratio, and the number of terms, to find the last term:—

RULE.

$$l = ar^{n-1}. \quad (1)$$

INTERPRETATION.—Multiply the first term by the common ratio raised to that power which is indicated by one less than the number of terms. The result will be the last term.

EXAMPLE 1.—What is the 9th term of the series 7, 21, 63, &c.?

OPERATION.

Here $a = 7$, $r = 3$, and $n = 9$.

Then $l = ar^{n-1} = 7 \times 3^{9-1} = 7 \times 3^8 = 7 \times 6561 = 45927$. *Ans.*

22. Given the first term, the common ratio, and the last term, to find the sum of the series:—

RULE.

$$s = \frac{rl - a}{r - 1}. \quad (VI.)$$

INTERPRETATION.—Subtract the first term from the product of the common ratio by the last term and divide the remainder by one less than the common ratio.

EXAMPLE 2.—The first term of a geometrical series is 5, the common ratio 4, and the last term 1000000. What is the sum of all the terms?

OPERATION.

Here $a = 5$, $r = 4$, and $l = 1000000$.

Then $s = \frac{rl - a}{r - 1} = \frac{4 \times 1000000 - 5}{4 - 1} = \frac{3999995}{3} = 1333331\frac{2}{3}$. *Ans.*

23. Given the first term, the common ratio and the number of terms, to find the sum of the series:—

RULE.

$$s = a \left(\frac{r^n - 1}{r - 1} \right) \quad (V.)$$

INTERPRETATION.—Find that power of the common ratio which is indicated by the number of terms, subtract one from it, and divide the remainder by one less than the common ratio.

Lastly, multiply the quotient thus obtained by the first term of the series, and the result will be the sum of all the terms.

EXAMPLE 3.—The first term of a geometrical series is 3, the common ratio is 4, and the number of terms 9. Required the sum of the series.

OPERATION

Here $a = 3$, $r = 4$, and $n = 9$.

$$\text{Then } s = a \left(\frac{r^n - 1}{r - 1} \right) = 3 \times \frac{4^9 - 1}{4 - 1} = 3 \times \frac{262144 - 1}{3} = 262143. \text{ Ans.}$$

24. To find the common ratio when the first term, the last term, and the sum of the terms are given:—

RULE.

$$r = \frac{s - a}{s - l} \text{ (XV)}$$

INTERPRETATION.—Divide the difference between the first term and the sum by the difference between the last term and the sum: the quotient will be the common ratio.

EXAMPLE 4.—The first term of a geometrical series is 1, the last term 19683, and the sum of all the terms 29524. What is the common ratio?

OPERATION.

Here $a = 1$, $l = 19683$, and $s = 29524$.

$$\text{Then } r = \frac{s - a}{s - l} = \frac{29524 - 1}{29524 - 19683} = \frac{29523}{9841} = 3. \text{ Ans.}$$

EXERCISE.

- A nobleman dying left 11 sons, to whom he bequeathed his property as follows: to the youngest he gave £1024; to the next, as much and a half: to the next, $1\frac{1}{2}$ of the preceding son's share; and so on. What was the eldest son's fortune; and what was the amount of the nobleman's property? *Ans.* The eldest son received £59049, and the father was worth £175099.
- The first term of a geometrical progression is 7, the last term is 1240029, and the sum of all the terms is 1860040. What is the ratio? *Ans.* 3.
- What debt can be discharged in a year by monthly payments in geometrical progression, the first term being £1, and the last £2048; and what will be the common ratio? *Ans.* The debt will be £4095; and the ratio 2.
- The ratio of the terms of a geometrical progression is $\frac{3}{2}$, the number of terms is 8, and the last term is $106\frac{1}{2}$. What is the sum of all the terms? *Ans.* $307\frac{1}{2}$.
- In a geometrical progression the first term is 1, the number of terms 7, and the common ratio 3, what is the sum of the series? *Ans.* 1093.

10. The first term of a geometrical progression is 1, the last term is 10077696, and the number of terms is 10. What is the sum of all the terms? *Ans.* 12093235.
11. The first term of a geometrical progression is 6, the last term is 3072, and the sum of all the terms is 6138. What is the ratio? *Ans.* 2.
12. The ratio of the terms of a geometrical progression is 2, the number of terms is 11, and the sum of all the terms is 20470. What is the last term? *Ans.* 10240.
13. A gentleman married his daughter on New Year's day, and gave her husband 1 shilling towards her portion, and was to double it on the first day of every month during the year. What was her portion? *Ans.* £204 15s.
13. What will be the price of a horse sold for 1 farthing for the first nail in his shoes, 2 farthings for the second, 4 for the third, &c., allowing 8 nails in each shoe? *Ans.* £4473924 5s. 3½d.
15. The first term of a geometrical progression is 4, the last term is 78732, and the number of terms is 10. What is the ratio? *Ans.* 3.
16. A person travelling, goes 5 miles the first day, 10 miles the second day, 20 miles the third day, and so on, increasing in geometrical progression. If he continue to travel in this way for 7 days, how far will he go the last day? *Ans.* 320 miles.
17. The first term of a geometrical progression is 5, the last term is 327680, and the ratio is 4. What is the sum of all the terms? *Ans.* 436905.
18. A king in India, named Sheran, wished (according to the Arabic author Asephad,) that Sessa, the inventor of chess, should himself choose a reward. He requested the king to give him 1 grain of wheat for the first square, 2 grains for the second square, 4 grains for the third square, and so on; reckoning for each of the 64 squares of the board twice as many grains as for the preceding. Sheran was angry at a demand apparently so insignificant; but when it was calculated, to his astonishment it was found to be an enormous quantity. What was the number of grains of wheat and what was its worth at \$1·50 per bushel, reckoning 7680 grains to a pint?
Ans. 18446744073709551615 grains.
37529996894754 bushels.
\$56294995342131.
19. The ratio of the terms of a geometrical progression is 3, the number of terms is 10, and the sum of all the terms is 295240. What is the last term? *Ans.* 196830.

20. The first term of a geometrical progression is 1, the last term is 2048, and the number of terms is 12. What is the sum of all the terms? *Ans.* 4095.
21. The first term of a geometrical progression is 5, the ratio is 4, and the number of terms 9. What is the last term? *Ans.* 327680.

25. When the common ratio of a geometrical series is a proper fraction, i. e., less than 1, the series is a descending one, and when the number of terms n becomes very large r^n becomes very small. In an infinite descending series r^n becomes infinitely small, i. e. its value becomes $= 0$, and therefore ar^n may be neglected and the formula for finding the sum becomes $s = \frac{ar^n - a}{r - 1} = \frac{-a}{r - 1} = \frac{a}{1 - r}$. Hence for finding the sum of any infinite series when r is less than 1:—

RULE.

$$s = \frac{a}{1 - r} \quad (\text{xxi})$$

INTERPRETATION.—The sum of an infinite series is found by dividing the first term by unity minus the common ratio.

EXAMPLE 22.—What is the sum of the infinite series $1 + \frac{1}{6} + \frac{1}{36} + \frac{1}{216}$, &c.?

OPERATION.

Here $a = 1$ and $r = \frac{1}{6}$

Then $s = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{6}} = \frac{1}{\frac{5}{6}} = \frac{6}{5} = 1\frac{1}{5}$. *Ans.*

EXAMPLE 23.—What is the sum of the infinite series $\cdot 734$?

OPERATION.

Here $a = \frac{734}{1000}$ and $r = \frac{1}{1000}$.

Then $s = \frac{a}{1 - r} = \frac{\frac{734}{1000}}{1 - \frac{1}{1000}} = \frac{734}{999} = 734\frac{4}{999}$. *Ans.*

EXERCISE.

24. What is the sum of the infinite series $\frac{2}{7}, \frac{6}{49}, \frac{18}{343}$, &c.? *Ans.* $\frac{5}{7}$.
25. What is the sum of the infinite series $4, 2, 1, \frac{1}{2}, \frac{1}{4}$, &c.? *Ans.* 8.
26. What is the sum of the infinite series $\cdot 79$? *Ans.* $79\frac{1}{9}$.
27. What is the sum of the infinite series $\cdot 1234$? *Ans.* $1234\frac{4}{999}$.

26. To insert any number of means between two given extremes :

RULE.

If the series is an arithmetical one, find the common difference by formula IX. ART. 8. Then add this common difference to the first term and the result will be the second term; add the common difference to the second and the result will be the third term, &c.

If the series is a geometrical one, find the common ratio by formula XIII. ART. 20. Then multiply the first term by the common ratio and the product will be the second term; multiply the second term by the common ratio and the result will be the third, &c.

EXAMPLE 1.—Insert 7 arithmetical means between 3 and 51?

OPERATION.

Since there are 7 means and 2 extremes the number of terms is 9.

$$\text{Then } d = \frac{l - a}{n - 1} = \frac{51 - 3}{9 - 1} = \frac{48}{8} = 6.$$

1st term = 3; 2nd = 3 + 6 = 9; 3rd = 9 + 6 = 15; 4th = 15 + 6 = 21;
5th = 21 + 6 = 27; 6th = 27 + 6 = 33, and so on.

And series is 3, 9, 15, 21, 27, 33, 39, 45, 51.

EXAMPLE 2.—Insert 6 geometrical means between 1 and 128?

OPERATION.

Since there are 6 means and 2 extremes the number of terms is 8.

$$\text{Then } r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = \left(\frac{128}{1}\right)^{\frac{1}{8-1}} = (128)^{\frac{1}{7}} = 2.$$

Hence 2nd term = $1 \times 2 = 2$; 3rd term = $2 \times 2 = 4$; 4th = $4 \times 2 = 8$, &c.

And series is 1, 2, 4, 8, 16, 32, 64, 128.

EXERCISE.

3. Insert 9 arithmetical means between 2 and 92.

Ans. 2, 11, 20, 29, 38, 47, 56, 65, 74, 83, 92.

4. Insert 4 arithmetical means between 7 and 50.

Ans. 7, 15 $\frac{1}{2}$, 24 $\frac{1}{2}$, 32 $\frac{1}{2}$, 41 $\frac{1}{2}$, 50.

5. Find 8 geometrical means between 4096 and 8.

Ans. 2048, 1024, 512, 256, 128, 64, 32, and 16.

6. Find 7 geometrical means between 14 and 23514624.

Ans. 84, 504, 3024, 18144, 108864, 653184, and 3919104.

POSITION.

27. Position is a rule which enables us to solve, by means of assumed numbers, a class of problems which we could not otherwise solve without the aid of algebra.

NOTE.—Position is also called the Rule of False, or the Rule of Trial and Error.

28. Position is divided into:—

1st. Single Position—when only one assumed number is used.

2nd. Double Position—when two assumed numbers are used.

29. Single Position is employed in the solution of those problems in which the required number is increased or decreased in any given ratio, i. e., when it is increased or diminished by *any part of itself*, or when it is *multiplied or divided by any given number*.

30. Double Position is employed in the solution of those problems in which the *result* found by increasing or decreasing the required number in any given ratio, is itself increased or diminished by some other number which is no known part or multiple of the required number.

SINGLE POSITION.

31. Single Position proceeds upon the principle that the results are proportional to the numbers used, and is employed in all cases when the problem can be stated algebraically in the form of $ax = b$, where $x =$ the required number, a the given multiplier, integral or fractional, and b the given result.

32. Let it be required to find a value of x such that $ax = b$. Suppose x' to be this value, and instead of b we obtain b' for the result. Then we have $ax = b$ and $ax' = b'$, and dividing we get $\frac{ax'}{ax} = \frac{b'}{b}$ or $\frac{x'}{x} = \frac{b'}{b}$ whence $b' :$

$$b :: x' : x \text{ or } x = \frac{b}{b'} \times x'.$$

Hence for single position we deduce the following:—

RULE.

Assume a number, and perform with it the operations described in the question; then say, as the result obtained is to the number used, so is the true or given result to the number required.

EXAMPLE 1.—What number is that which being increased by its fourth part and diminished by its fifth part gives 63 for the result?

OPERATION.

Assume any number, 40.* Then *one-fourth* of number = 10, and *one-fifth* = 8.

* For the sake of convenience we assume a number of which we can take the required parts without using fractions.

$40 + 10 - 8 = 42$, which by the question should have been 63.

Then—Result obtained: Result required :: Number used: Number required.

$$\text{Or, } 42 : 63 :: 40 : \frac{63 \times 40}{42} = 60 \text{ Ans.}$$

PROOF.— $60 + \frac{1}{4}$ of $60 - \frac{1}{8}$ of $60 = 63$.

EXAMPLE 2.—A teacher being asked how many pupils he had, replied, if you add $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{6}$ of the number together, the sum will be 18; what was their number?

OPERATION.

Assume 60 to be the number of pupils.

Then one-third of 60 = 20

one-fourth of 60 = 15

one-sixth of 60 = 10

Sum = 45, but it should, by question, equal 18.

$$\text{Then } 45 : 18 :: 60 : \frac{18 \times 60}{45} = 24 \text{ Ans.}$$

PROOF.— $\frac{1}{3}$ of 24 + $\frac{1}{4}$ of 24 + $\frac{1}{6}$ of 24 = 18.

EXERCISE.*

3. A gentleman distributed 78 pence among a number of poor persons, consisting of men, women, and children; to each man he gave 6d., to each woman 4d., to each child 2d.; there were twice as many women as men, and three times as many children as women. How many were there of each? *Ans.* 3 men, 6 women, and 18 children.
4. A person bought a chaise, horse, and harness, for £60; the horse came to twice the price of the harness, and the chaise to twice the price of the horse and harness. What did he give for each? *Ans.* He gave for the harness, £6 13s. 4d.; for the horse, £13 6s. 8d.; and for the chaise, £40.
5. A's age is double that of B's; B's is treble that of C's; and the sum of all their ages is 140. What is the age of each? *Ans.* A's is 84, B's 42, and C's 14.
6. After paying away $\frac{1}{4}$ of my money, and then $\frac{1}{6}$ of the remainder, I had 72 guineas left. What had I at first? *Ans.* 120 guineas.

*All questions in position may be solved by simple analysis, and very frequently this is the better method, and indeed the teacher should insist upon the pupil thus solving each problem. The following will serve as examples of the mode of solution.

EXAMPLE 5.—Since 140 is equal to A's age, + B's age, + C's age, and B's age is equal to three times C's, and A's to 6 times C's, it follows that 140 is equal to $1 + 3 + 6 = 10$ times C's age, and hence C's age is $\frac{1}{10}$ of $140 = 14$; 's = $14 \times 3 = 42$; and A's = $14 \times 6 = 84$.

7. A can do a piece of work in seven days ; B can do the same in 5 days ; and C in 6 days. In what time will all of them execute it ? *Ans.* In $1\frac{9}{17}$ days.
8. A and B can do a piece of work in 10 days ; A by himself can do it in 15 days. In what time will B do it ? *Ans.* In 30 days.
9. A cistern has three pipes ; when the first is opened all the water runs out in one hour ; when the second is opened, it runs out in two hours ; and when the third is opened, in three hours. In what time will it run out, if all the pipes are kept open together ? *Ans.* In $\frac{5}{11}$ hours.
10. What is that number whose $\frac{1}{3}$, $\frac{1}{6}$, and $\frac{1}{7}$ parts, taken together, make 27 ? *Ans.* 42.
11. There are 5 mills ; the first grinds 7 bushels of corn in 1 hour, the second 5 in the same time, the third 4, the fourth 3, and the fifth 1. In what time will the five grind 500 bushels, if they work together ? *Ans.* In 25 hours.
12. There is a cistern which can be filled by a pipe in 12 hours ; it has another pipe in the bottom, by which it can be emptied in 18 hours. In what time will it be filled, if both are left open ? *Ans.* In 36 hours.

DOUBLE POSITION.

33. When the number sought is to be increased or diminished by some *absolute* number, which is not a known multiple, or part of it—or when *two* propositions, neither of which can be banished, are contained in the problem, we use *double position*, assuming *two* numbers. If the number sought is, during the process indicated by the question, to be involved or evolved, we obtain only an *approximation* to the quantity required. In other words *double position* is employed in all cases in which the problem stated algebraically would take the form of

$$ax + b = c$$

where x is the number sought, a the given multiplier, integral or fractional, b the given increment, and c the given result.

EXAMPLE 7. BY ANALYSIS.—Since A can do the whole work in 7 days, in 1 day he will do $\frac{1}{7}$ of the whole work, similarly in 1 day B will do $\frac{1}{6}$, and C $\frac{1}{5}$ of the whole work. Therefore working together they will do $\frac{1}{7} + \frac{1}{6} + \frac{1}{5} = \frac{47}{210}$ of the whole work, and they will require as many days to do the whole work as $\frac{210}{47}$ is contained times in 1, i. e., $1 \div \frac{47}{210} = 1\frac{10}{47}$ days *Ans.*

OPERATION.

Suppose he worked 50 days; then he was idle 30 days.

| | | |
|--|-----------------|-----------|
| Sum earned = $50 \times 75 = \$37.50$ | True result | = \$25.00 |
| Sum forfeited = $30 \times 50 = 15.00$ | Result obtained | = 22.50 |
| Sum received = 22.50 | Error | = 2.50 |

Again suppose he worked 40 days; then he lost 40 days.

| | | |
|--|-----------------|-----------|
| Sum earned = $40 \times 75 = \$30.00$ | Result required | = \$25.00 |
| Sum forfeited = $40 \times 50 = 20.00$ | Result obtained | = 10.00 |
| Sum received = 10.00 | Error | = 15.00 |

| Errors. | × | Assumed numbers. | = | Products. |
|---------|---|------------------|---|-----------|
| -15 | × | 50 | = | 750 |
| -2½ | × | 40 | = | 100 |

Difference of errors = $12\frac{1}{2}$. Difference of products = 650

Therefore result required = $650 \div 12\frac{1}{2} = 52$ days.

Number of idle days = $80 - 52 = 28$. *Ans.*

PROOF.—Sum earned = $52 \times 75 = \$39.00$

Sum forfeited = $28 \times 50 = 14.00$

Sum received = \$25.00.

EXAMPLE 3.—What number is that which, being multiplied by 3, the product increased by 4, and that sum divided by 8, the quotient shall be 32?

OPERATION.

Assume 40 to be the number.

Then $40 \times 3 = 120 + 4 = 124 \div 8 = 15\frac{1}{2}$ = result obtained.

32 = result required.

Error = $-16\frac{1}{2}$

Again: assume 100 to be the number.

Then $100 \times 3 = 300 + 4 = 304 \div 8 = 38$ = result obtained.

32 = result required.

Error = +6

| Errors. | × | Assumed numbers. | = | Products. |
|---------|---|------------------|---|-----------|
| -10½ | × | 100 | = | 1650 |
| + 6 | × | 40 | = | 240 |

Sum of error = $22\frac{1}{2}$.

Sum of products = 1890

Required number = $\frac{1890}{22\frac{1}{2}} = 84$. *Ans.*

PROOF.— $84 \times 3 = 252 + 4 = 256 \div 8 = 32$.

NOTE.—In this example we take the sum of the errors for a divisor and the sum of the products for a dividend, because the errors are not both plus or both minus.

EXAMPLE 4.—What is that number which is equal to 4 times its square root + 21 ?

OPERATION.

Assume 64

$$\begin{array}{r}
 \sqrt{64} = 8 \\
 \underline{\quad 4} \\
 32 \\
 \underline{21} \\
 \hline
 53, \text{ result obtained.} \\
 64, \text{ result required.} \\
 \hline
 -11, \text{ difference.} \\
 81 \\
 \hline
 891
 \end{array}$$

Assume 81

$$\begin{array}{r}
 \sqrt{81} = 9 \\
 \underline{\quad 4} \\
 36 \\
 \underline{21} \\
 \hline
 57, \text{ result obtained.} \\
 81, \text{ result required.} \\
 \hline
 -24, \text{ difference.} \\
 64 \\
 \hline
 1536 \\
 891
 \end{array}$$

13)645

The first approximation is 49'6154.

It is evident that 11 and 24 are not the errors in the assumed numbers multiplied or divided by the same quantity, and, therefore, as the reason upon which the rule is founded, does not apply, we obtain only an approximation. Substituting this, however, for one of the assumed numbers, we obtain a still nearer approximation.

SECOND RULE.

Find the errors by the last rule; then divide their difference (if they are both of the same kind), or their sum (if they are of different kinds), into the product of the difference of the numbers and one of the errors. The quotient will be the correction of that error which has been used as multiplier.

NOTE.—This rule depends upon the principle that the difference between the assumed numbers and the true numbers are proportional to the differences of the results obtained using the assumed numbers and that given in the problem. As in the last rule, when the question could not be resolved by an equation of the first degree, the rule gives only an approximation to the correct result.

EXAMPLE 5.—If to four times the price of my horse £10 be added the result will be £100. What is the price of my horse ?

OPERATION.

Assume £19, and secondly £25 as the price of the horse—

| | |
|----------------------------|----------------------------|
| Then 19 | 25 |
| <u>4</u> | <u>4</u> |
| 76 | 100 |
| <u>10</u> | <u>10</u> |
| 86, the result obtained. | 110, the result obtained. |
| 100, the result required. | 100, the result required. |
| -14 is an error of defect. | +10 is an error of excess. |

The errors are of *different* kinds: and their *sum* is $14+10=24$; and the difference of the assumed numbers is $25-19=6$. Therefore

14, one of the errors,
is multiplied by $\frac{14}{6}$, the difference of the numbers. Then divide by

$$\frac{24}{6}8\frac{1}{3}$$

and $3\frac{1}{3}$ is the correction for 19, the number which gave an error of 14.

$19+$ (the error being one of *defect*, the correction is to be *added*) $3\frac{1}{3}=22\frac{2}{3}$
 $=£22$ 10s. is the required quantity.

EXERCISE.

6. A son asked his father how old he was, and received the following answer. Your age is now $\frac{1}{4}$ of mine, but 5 years ago it was only $\frac{1}{5}$. What are their ages?
Ans. 80 and 20.
7. Required what number it is from which, if 34 be taken, 3 times the remainder will exceed it by $\frac{1}{2}$ of itself? *Ans.* 58 $\frac{2}{3}$.
8. A and B go out of a town by the same road. A goes 8 miles each day; B goes 1 mile the first day, 2 the second, 3 the third, &c. When will B overtake A?

| Suppose | A. | B. | Suppose | A. | B. |
|---------|------|----|---------|------|----|
| | 5 | 1 | | 7 | 1 |
| | 8 | 2 | | 8 | 2 |
| | — | 3 | | — | 3 |
| | 10 | 4 | | 56 | 4 |
| | 15 | 5 | | 28 | 5 |
| | — | — | | — | 6 |
| | 5)25 | 15 | | 7)28 | 7 |
| | — | | | — | — |
| | —5 | | | —4 | 28 |
| | 7 | | | —5 | |
| | — | | | — | |
| | 35 | | | — | |
| | 20 | | | — | |
| | — | | | — | |
| | 1)15 | | | — | |

$5-4=1=$ difference of errors.

We divide the entire error by the number of days in each case, which gives the error in one day.

9. What are those numbers which, when added, make 25; but when one is halved and the other doubled, give equal results?
Ans. 20 and 5.
10. Two contractors, A and B, are each to build a wall of equal dimensions; A employs as many men as finish $22\frac{1}{2}$ perches in a day; B employs the first day as many as finish 6 per., the second as many as finish 9, the third as many as finish 12, &c. In what time will they have built an equal number of perches?
Ans. 12 days.
11. What is the number whose $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ multiplied together, make 24?

| Suppose 12 | Suppose 4 |
|------------------------------|------------------------------|
| $\frac{1}{2} = 6$ | $\frac{1}{2} = 2$ |
| $\frac{1}{4} = 3$ | $\frac{1}{4} = 1$ |
| Product = 18 | Product = 2 |
| $\frac{2}{3} = 4\frac{1}{2}$ | $\frac{2}{3} = 1\frac{1}{2}$ |
| 81, result obtained. | 3, result obtained. |
| 24, result required. | 24, result required. |
| +57, error. | -21, error. |
| 64, the cube of 4. | 1728, the cube of 12. |
| 3648, product. | 36288 to this product. |
| 57+21=78 | 3648 is added. |
| | 78)38936 is the sum. |

And 512 the quotient.

$\sqrt[3]{512} = 8$, is the required number.

We multiply the alternate error by the *cube* of the supposed number, because the error belongs to $\frac{3}{4}$ part of the *cube* of the assumed numbers and not to the numbers themselves: for in reality it is the cube of some number that is required—since 8 being assumed, according to the question we have $\frac{8}{2} \times \frac{8}{4} \times \frac{8 \times 8}{8} = 24$; or $\frac{3}{64} \times 8^3 = 24$.

12. What number is it whose $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, and $\frac{1}{8}$, multiplied together, will produce 6998 $\frac{2}{3}$? *Ans.* 36.
13. A said to B give me one of your shillings and I shall have twice as many as you will have left. B answered, if you give me one shilling I shall have as many as you. How many had each? *Ans.* A 7 and B 5.
14. There are two numbers which, when added together, make 30; but the $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of the greater are equal to $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$ of the lesser. What are they? *Ans.* 12 and 18.
15. A gentleman has 2 horses and a saddle worth £50. The saddle, if set on the back of the first horse, will make his value double that of the second; but if set on the back of the second horse, will make his value treble that of the first. What is the value of each horse? *Ans.* £30 and £40.
16. A gentleman finding several beggars at his door, gave to each 4d. and had 6d. left, but if he had given 6d. to each he would have 12d. too little. How many beggars were there? *Ans.* 9.

COMPOUND INTEREST.

35. Let P = the principal, I = the interest, A = the amount, t = the number of payments, and r = the rate per unit for one payment.

Then since r is the interest of \$1 for one payment, the amount of \$1 for one payment is $1+r$, and since the principal is always proportional to the amount.

$$1 : 1+r :: P : P(1+r) = \text{Amount of } P \text{ at end of 1st period,}$$

$$1 : 1+r :: P(1+r) : P(1+r)^2 = \text{Amount of } P \text{ at end of 2nd period.}$$

$$1 : 1+r :: P(1+r)^2 : P(1+r)^3 = \text{Amount of } P \text{ at end of 3rd period,}$$

$$1 : 1+r :: P(1+r)^3 : P(1+r)^4 = \text{Amount of } P \text{ at end of 4th period.}$$

And so on; hence at the end of the t^{th} period $A = P(1+r)^t$ which is

$$A = P(1+r)^t \text{ (I)}$$

$$P = \frac{A}{(1+r)^t} \text{ (II)}$$

$$r = \sqrt[t]{\frac{A}{P}} - 1 \text{ (III)}$$

$$t = \frac{\log. A - \log. P}{\log. (1+r)} \text{ (IV)}$$

$$t = \frac{\log. n}{\log. (1+r)} \text{ (V)}$$

formula (I) in the margin.

Dividing each side of (I) by $(1+r)^t$ we get formula (II) in the margin.

Dividing each side of (I) by P we get $(1+r)^t$

$= \frac{A}{P}$ extracting the t^{th} root, and transposing the 1, we get formula (III).

Obtaining as before $(1+r)^t = \frac{A}{P}$ and applying

the principle of logarithms we get $\log. (1+r)$

$\times t = \log. A - \log. P$, and dividing each side

by $\log. (1+r)$ we get $t = \frac{\log. A - \log. P}{\log. (1+r)}$

Which is (IV) of the margin.

Lastly, to find the time in which any sum of

money will amount to n times itself at a given

rate per cent. compound interest we substitute

nP for A in formula (I.) which gives us nP

$= P(1+r)^t$ and dividing each of these by

P we get $n = (1+r)^t$ whence $\log. n = \log.$

$(1+r) \times t$; or $t = \frac{\log. n}{\log. (1+r)}$ which is for-

mula (V.)

APPLICATIONS.

When the principal, rate per cent., and time are given to find the amount:—

RULE.

$$A = P(1+r)^t \text{ or } \log. A = \log. P + \log. (1+r) \times t.$$

INTERPRETATION.—Multiply the logarithm of the amount of \$1 for one payment, by the number of payments, and to the product add the logarithm of the principal; the result will be the logarithm of the amount.

II. Find the natural number corresponding to this logarithm and the result will be the answer.

EXAMPLE 1.—To what sum will \$750 amount in 3 years, at 2 per cent. quarterly compound interest?

OPERATION.

Here $P = 750$, $r = .02$ and $t = 12$, since there are 12 quarters in 3 years.

Then $A = P(1+r)^t$ or $\log. A = \log. P + \log. (1+r) \times t = 2.875061 + 0.008600 \times 12 = 2.978261 = \log.$ of Answer. Hence amount = \$951.17.

36. When the amount, rate, and time are given to find the principal :—

RULE.

$$P = \frac{A}{(1+r)^t}; \text{ or } \log. P = \log. A - \log. (1+r) \times t.$$

INTERPRETATION.—Take the number expressing the amount of \$1 for one payment, and raise it to the power indicated by the number of payments.

II. Divide the given amount by the number thus obtained and the quotient will be the required principal.

BY LOGARITHMS.

Take the logarithm of the amount of \$1 for one payment, and multiply it by the number of payments.

Subtract the logarithm thus obtained from the logarithm of the given amount; the remainder will be the logarithm of the required principal.

EXAMPLE 2.—What principal put out at compound interest, at the rate of $3\frac{1}{2}$ per cent. half yearly, will amount to \$8764.00 in 11 years?

OPERATION.

Here $A = 8764$, $r = .035$ and $t = 22$.

Then $P = \frac{A}{(1+r)^t}$ or $\log. P = \log. A - \log. (1+r) \times t$.

$\log. P = 13.942702 - 0.014940 \times 22 = 3.942702 - 0.328680 = 3.614022$.

Hence $P = \$4111.70$. *Ans.*

37. When the amount, principal, and time are given to find the rate per cent. :—

RULE.

$$r = t \sqrt[t]{\left(\frac{A}{P}\right)} - 1; \text{ or } \log. (1+r) = \frac{\log. A - \log. P}{t}$$

INTERPRETATION.—Divide the amount by the principal and extract that root of the quotient which is indicated by the number of payments.

II. Subtract 1 from the root thus obtained and the remainder will be the rate per unit, multiply this by 100 and the result will be the rate per cent.

BY LOGARITHMS.

Subtract the logarithm of the principal from the logarithm of the given amount, and divide the difference by the number of payments; the result will be the logarithm of the amount of \$1 for one payment.

Find the natural number corresponding to this, and from it subtract 1, the result will be the rate per unit, and this multiplied by 100 gives the rate per cent.

EXAMPLE 3.—At what rate per cent. compound interest, payable half-yearly, will \$278 amount to \$6742 in 27 years?

OPERATION.

Here $A = 6742$, $P = 278$ and $t = 54$.

Then $\log. (1+r) = \frac{\log. A - \log. P}{t} = \frac{3.828789 - 2.444045}{54} = \frac{1.384744}{54}$
 $= .0256434$. Hence $1+r = 1.06$, $r = .06$, and rate per cent. = 6. *Ans.*

38. When the amount, principal and rate are given to find the time:—

RULE.

$$t = \frac{\log. A - \log. P}{\log. (1+r)}$$

INTERPRETATION.—*Subtract the logarithm of the principal from the logarithm of the given amount, and divide the remainder by the logarithm of the amount of \$1 for one payment; the quotient will be the number of the payments.*

EXAMPLE 4.—In what time will \$729 amount to \$7143 at 2½ per cent. compound interest, quarterly?

OPERATION.

Here $A = 7143$, $P = 729$ and $r = .025$.

Then $t = \frac{\log. A - \log. P}{\log. (1+r)} = \frac{3.853881 - 2.862728}{0.010724} = \frac{0.991153}{0.010724} = 92.42$ pay-
 ments = 23.105 years = 23 years, 1 month, 7.8 days. *Ans.*

39. To find in what time any sum of money will amount to n times itself at any given rate per cent. compound interest:—

RULE.

$$t = \frac{\log. n}{\log. (1+r)}$$

INTERPRETATION.—*Find the logarithm of the number expressing to how many times itself the given sum is to amount, and divide it by the logarithm of the amount of \$1 for one payment; the result will be the required time.*

EXAMPLE 5.—In what time will any sum of money amount to five times itself at 5 per cent. per annum, compound interest?

OPERATION.

Here $n = 5$ and $r = .05$.

Then $t = \frac{\log. n}{\log. (1+r)} = \frac{0.698970}{0.021189} = 32.987$ yrs. = 32 years 11 months 25 days. *Ans.*

EXAMPLE 6.—In what time will any sum of money amount to nine times itself at 3½ per cent. quarterly, compound interest?

OPERATION.

Here $n = 9$ and $r = .035$.

Then $t = \frac{\log. n}{\log. (1+r)} = \frac{0.954243}{0.014940} = 63.8716$ payments = 15.9679 years = 15 years 11 months 18 days. *Ans.*

EXERCISE.

7. What is the amount and compound interest of \$713.29 for 7 years at $4\frac{1}{2}$ per cent. half yearly?

Ans. Amount = \$1320.96.

Compound interest = \$ 607.67.

8. In what time will any sum of money amount to seven times itself at $1\frac{1}{2}$ per cent. quarterly, compound interest?

Ans. 32 years 8 months 2 days.

9. In what time will \$111.11 amount to \$1111.11 at 8 per cent. per annum, compound interest? *Ans.* 29 years 11 mos.

10. At what rate per cent. quarterly will \$222.22 amount to \$3333.33 in 30 years, compound interest being allowed?

Ans. $2\frac{1}{2}\%$.

11. In what time will any sum of money double itself at 7 per cent. per annum, compound interest?

Ans. 10 years 2 months 28 days.

12. What principal put out at compound interest at the rate of $2\frac{1}{2}$ per cent. quarterly will amount to \$100 in 7 years?

Ans. \$53.63.

13. To what sum will \$2468.13 amount in 13 years at compound interest $3\frac{1}{2}$ per cent. half yearly? *Ans.* \$6427.705.

14. What principal will amount to \$7137.40 in 11 years, compound interest at the rate of $4\frac{1}{2}$ per cent. half yearly being allowed? *Ans.* \$2856.723.

15. In what time will any sum of money amount to 19 times itself at $5\frac{1}{2}$ per cent. half yearly, compound interest?

Ans. 28 years 9 months 8 days.

ANNUITIES.

40. An Annuity is any periodical income payable at equal intervals, as yearly, half yearly, quarterly, &c.

41. An Annuity *in possession* is one that is entered upon already.

42. An Annuity *in reversion* or a *deferred annuity* is one whose first payment is not to be made until after the expiration of a given time or until the occurrence of a specified event.

43. An Annuity *certain* is one that is to continue for a fixed number of years.

44. An Annuity *contingent* or a *life annuity* is one that is to continue to be paid only so long as one or more individuals shall live.

45. A *Perpetuity* is an annuity that is to continue for ever.

46. An Annuity is in *arrears* when one or more payments are retained after they have become due.

47. The *amount of an annuity* is the sum of the payments forborne (i.e. in arrears) and the whole interest due upon them.

48. The *present worth of an annuity* is that sum which, being put out at interest until the annuity ceases, would produce a sum equal to what would have been accumulated had the annuity been left unpaid until that time.

49. Annuities are calculated at both simple and compound interest.

ANNUITIES AT SIMPLE INTEREST.

50. Let a = a single payment of the annuity, t = number of payments, r = rate per unit for one period, and A = amount of the annuity.

Then when the annuity is forborne any number of payments, the last payment being made at the time it falls due, is equal to a ; last payment but one = a + interest on a for one period = $a + ar$; last but two = a + interest on a for two payments = $a + 2ar$; last but three = $a + 3ar$; last but four = $a + 4ar$, &c.; and hence the last payment = a + interest on a for one less than the number of payments = $a + (t-1)ar$.

Hence the payments forborne, with their interest, constitute a series in arithmetical progression where the first term is a , the last term $a + (t-1)ar$, the common difference ar , the sum of the series A , and the number of terms t .

Then (Art. 5.) $A = a + (a + ar) + (a + 2ar) + (a + 3ar)$, &c. + $\{a + (t-1)ar\}$

Whence (Art. 6.) $A = \{a + (a + (t-1)ar)\} \frac{t}{2} = (1 + \frac{(t-1)r}{2})ta$ which is formula I. in the margin.

$$A = at(1 + \frac{(t-1)r}{2}) \quad (\text{I.})$$

$$a = \frac{2A}{t(2 + (t-1)r)} \quad (\text{II.})$$

$$r = \frac{2(A - at)}{at(t-1)} \quad (\text{III.})$$

$$t = \sqrt{\frac{8rA}{a} + (2-r)^2} - (2-r) \quad (\text{IV.})$$

Formulas II., III., and IV., are derived from formula I., by transposition, &c.

No general formula has yet been discovered for the summation of a series for finding the *present value* of an annuity at simple interest. The rule generally adopted for finding the present value of an annuity at simple interest is the following:—

Find the present worth of each payment by itself, discounting from the time it falls due—the sum of the present worth of all the payments will be the present worth of the annuity.

NOTE.—The absolute absurdity of purchasing annuities by simple interest is evident from the fact that the interest of the sum required to purchase an annuity, discounting at 5 per cent. simple interest, actually exceeds the annuity; i. e., to purchase an annuity to continue only a limited number of years, requires a sum which will yield a larger yearly interest for ever. Hence the various rules given for finding the present value of annuities at simple interest are, in effect, valueless.

APPLICATIONS.

51. When the annuity, number of payments forborne, and the rate per cent. of interest are given, to find the amount:—

RULE.

$$A = at \left\{ 1 + \frac{(t-1)r}{2} \right\}$$

INTERPRETATION.—Multiply the rate per unit by one less than the number of payments and to half the result add 1.

• Multiply the number thus obtained by the product of the annuity by the number of payments and the result will be the required amount.

EXAMPLE 1.—If a pension of \$600 per annum be forborne 5 years, to what sum will it amount at 4 per cent. simple interest?

OPERATION.

Here $a = 600$, $t = 5$, $r = .04$.

Then $A = at \left\{ 1 + \frac{(t-1)r}{2} \right\} = 600 \times 5 \left\{ 1 + \frac{(5-1) \times .04}{2} \right\} = 3000 \times (1 + .08) = 3000 \times 1.08 = \3240 . *Ans.*

52. When the amount of the annuity forborne, the number of payments forborne, and the rate per cent. of interest allowed, are given, to find the annuity:—

RULE.

$$a = \frac{2A}{t \left\{ 2 + (t-1)r \right\}}$$

INTERPRETATION.—Multiply the rate per unit by one less than the number of payments and to the product add 2.

Multiply this sum by the number of payments, and divide twice the given amount of the annuity by the product thus obtained; the result will be the annuity required.

EXAMPLE 2.—What annuity payable quarterly, will amount to \$3625.25 in 7 years, at $4\frac{1}{2}$ per cent. per annum, simple interest?

OPERATION.

Here since the rate is $4\frac{1}{2}$ per cent. per annum or .045 per unit per annum, the rate per quarter = $.045 \div 4 = .01125$.

Then $t = 28$, $A = \$3225.25$ and $r = .01125$.

$$a = \frac{2A}{t\{2 + (t-1)r\}} = \frac{3225.25 \times 2}{28\{2 + (28-1) \times .01125\}} = \frac{6450.50}{28 \times (2 + .30375)}$$

$$= \frac{6450.50}{28 \times 2.30375} = \frac{6450.50}{64.505} = \$100 = \text{quarterly payment, and hence annual annuity} = \$400. \text{ Ans.}$$

53. The application and interpretation of the remaining formulæ will be readily understood from the foregoing examples.

EXERCISE.

3. In what time will an annuity of \$1000 per annum, payable half-yearly, amount to \$8365, allowing simple interest, at the rate of 6 per cent. per annum? *Ans.* 14 payments, or 7 years.

NOTE.—In this question we use formula IV, r being equal to .03 and $a = 500$.

4. If a rent of \$450 per annum, payable quarterly, be forborne for 11 years, to what does it amount, allowing 6 per cent. per annum simple interest? *Ans.* \$6546.37½.

NOTE.—Take $a = \$112.50$, $r = .015$ and $t = 44$.

5. At what rate per cent. per annum, simple interest, will an annuity of \$300, payable yearly, amount to \$1680 in 5 years? *Ans.* 6 per cent.
6. The rent of a farm is forborne for 8 years, and then amounts to \$2080. Now assuming the rent to be paid half-yearly, and simple interest at the rate of 8 per cent. per annum allowed, what was the rent of the farm? *Ans.* \$200.

ANNUITIES AT COMPOUND INTEREST.

54. Let A , a , r , t = same quantities as in last articles and also let v = present value of the annuity.

Then, as before, the last payment of a forborne annuity being paid when due, = a ; last payment but one, = a + interest of a for one payment = $a + ar = a(1+r)$ so also last payment but two, = $a(1+r)^2$; last but three = $a(1+r)^3$ &c., and last payment = $a(1+r)^{t-1}$.

Hence A , the amount of the annuity = $a + a(1+r) + a(1+r)^2 + a(1+r)^3 + \&c. + a(1+r)^{t-1}$ which is a geometrical series and is equal (Art. 18.)

$$A = \frac{a \{ (1+r)^t - 1 \}}{r} \quad (\text{I})$$

$$a = \frac{Ar}{(1+r)^t - 1} \quad (\text{II})$$

$$r = t \sqrt{\frac{Ar+a}{a}} - 1. \quad (\text{III})$$

$$t = \frac{\log. (Ar+a) - \log. a}{\log. (1+r)} \quad (\text{IV})$$

$$v = \frac{a}{r} \left\{ 1 - \frac{1}{(1+r)^t} \right\} \quad (\text{V})$$

$$a = \frac{vr(1+r)^t}{(1+r)^t - 1} \quad (\text{VI})$$

$$t = \frac{\log. a - \log. (a - vr)}{\log. (1+r)} \quad (\text{VII})$$

$$v = \frac{a}{r} \left\{ \frac{1}{(1+r)^t} - \frac{1}{(1+r)^{s+t}} \right\} \quad (\text{VIII})$$

$$v = \frac{a}{r} \quad (\text{IX})$$

$$a = vr \quad (\text{X})$$

$$r = \frac{a}{v} \quad (\text{XI})$$

$$v = \frac{a}{r(1+r)^t} \quad (\text{XII})$$

Formulas X and XI are derived from IX.

The present value of a freehold estate to a person to whom it will revert after s years and then continue for ever, is found from formula VIII and is represented by formula XII in the margin.

55. To facilitate the calculation of annuities the following tables are given, the first showing the amount of an annuity of \$1 at compound interest, and the second, the present value of an annuity of \$1 at compound interest.

$\frac{a \{ (1+r)^t - 1 \}}{r}$, which is formula I of margin.

Formulas II, III, and IV are obtained from formula I by transposition, &c.

Since the present value of an annuity at compound interest is that principal which put out at compound interest for the given time, would produce the amount of the annuity we have from Art. 36, formula I, $v, (1+r)^t = A = \frac{a \{ (1+r)^t - 1 \}}{r}$ whence by di-

viding by $(1+r)^t$, we get formula V in the margin.

Formulas VI and VII are derived from V.

To find the present value of an annuity which is to commence after t years and then continue for s years, we have from formula V, v for $s+t$ years, =

$$\frac{a \{ (1+r)^{s+t} - 1 \}}{r (1+r)^{s+t}}$$

$$\text{and for } t \text{ years alone, } v = \frac{a \{ (1+r)^t - 1 \}}{r (1+r)^t}$$

Therefore for t years to commence after s years. $v =$

$$\frac{a \{ (1+r)^{s+t} - 1 \}}{r (1+r)^{s+t}} - \frac{(1+r)^t - 1}{r (1+r)^t}$$

$$\text{or } v = \frac{a}{r} \left\{ \frac{1}{(1+r)^t} - \frac{1}{(1+r)^{s+t}} \right\}$$

which is formula VIII in the margin.

When an annuity lasts for ever as in the case of landed property, $(1+r)^t$ in formula V becomes infinitely great, and therefore

$$\frac{1}{(1+r)^t} = \frac{1}{\infty} = 0 \text{ and the formula for finding the present value of a perpetuity is reduced to the form given in IX.}$$

TABLE OF THE AMOUNTS OF AN ANNUITY
OF \$1 OR £1.

| No. of Pay-ments. | 3 per cent. | 4 per cent. | 5 per cent. | 6 per cent. |
|-------------------|-------------|-------------|-------------|-------------|
| 1 | 1'00000 | 1'00000 | 1'00000 | 1'00000 |
| 2 | 2'03000 | 2'01000 | 2'05000 | 2'06000 |
| 3 | 3'09000 | 3'12100 | 3'15250 | 3'18360 |
| 4 | 4'18363 | 4'24644 | 4'31012 | 4'37162 |
| 5 | 5'30913 | 5'41632 | 4'52563 | 5'63706 |
| 6 | 6'46841 | 6'63297 | 6'89191 | 6'97532 |
| 7 | 7'66246 | 7'89829 | 8'14201 | 8'39384 |
| 8 | 8'89234 | 9'21423 | 9'54911 | 9'89747 |
| 9 | 10'15911 | 10'58279 | 11'02656 | 11'49131 |
| 10 | 11'46388 | 12'00611 | 12'57789 | 13'18079 |
| 11 | 12'80970 | 13'48635 | 14'20679 | 14'97164 |
| 12 | 14'19203 | 15'02580 | 15'91713 | 16'86394 |
| 13 | 15'61779 | 16'62684 | 17'71298 | 18'85244 |
| 14 | 17'08632 | 18'29191 | 19'59863 | 21'01506 |
| 15 | 18'59891 | 20'02359 | 21'57856 | 23'27598 |
| 16 | 20'15688 | 21'82453 | 23'65749 | 25'67253 |
| 17 | 21'76139 | 23'69751 | 25'84037 | 28'21288 |
| 18 | 23'41443 | 25'64541 | 28'13238 | 30'90565 |
| 19 | 25'11687 | 27'67123 | 30'53900 | 33'75399 |
| 20 | 26'87037 | 29'77808 | 33'06595 | 36'75859 |
| 21 | 28'67618 | 31'96920 | 35'71925 | 39'92273 |
| 22 | 30'53678 | 34'24797 | 38'50521 | 43'25229 |
| 23 | 32'45288 | 36'61789 | 41'43047 | 46'75583 |
| 24 | 34'42647 | 39'08260 | 44'50200 | 50'43558 |
| 25 | 36'45926 | 41'64591 | 47'72710 | 54'28651 |
| 26 | 38'55304 | 44'31174 | 51'11345 | 58'31639 |
| 27 | 40'70963 | 47'08431 | 54'66931 | 62'53576 |
| 28 | 42'93092 | 49'96758 | 58'40258 | 68'02811 |
| 29 | 45'21885 | 52'96629 | 62'32271 | 73'63980 |
| 30 | 47'57541 | 56'08494 | 66'43885 | 79'45819 |
| 31 | 50'00268 | 59'32833 | 70'76079 | 84'80168 |
| 32 | 52'50276 | 62'70147 | 75'29829 | 90'88078 |
| 33 | 55'07784 | 66'20953 | 80'06377 | 97'34316 |
| 34 | 57'73018 | 69'85791 | 85'06396 | 104'18375 |
| 35 | 60'46208 | 73'65222 | 90'32631 | 111'43478 |
| 36 | 63'27504 | 77'59831 | 95'83623 | 119'12087 |
| 37 | 66'16742 | 81'70225 | 101'62814 | 127'26812 |
| 38 | 69'15045 | 85'97034 | 107'70954 | 135'89429 |
| 39 | 72'23423 | 90'40015 | 114'09502 | 145'05846 |
| 40 | 75'40126 | 95'02551 | 120'79977 | 154'76196 |
| 41 | 78'66339 | 99'82654 | 127'83976 | 165'04768 |
| 42 | 82'02320 | 104'81960 | 135'23175 | 175'95654 |
| 43 | 85'48389 | 110'01238 | 142'99334 | 187'50758 |
| 44 | 89'04841 | 115'41283 | 151'14300 | 199'75803 |
| 45 | 92'71986 | 121'02939 | 159'70015 | 212'74351 |
| 46 | 96'50116 | 126'87957 | 168'68516 | 226'50812 |
| 47 | 100'39650 | 132'94530 | 178'11924 | 241'09861 |
| 48 | 104'40839 | 139'26221 | 188'02539 | 256'56453 |
| 49 | 108'54065 | 145'83373 | 198'42666 | 272'95840 |
| 50 | 112'79687 | 152'66708 | 209'34790 | 290'33590 |

TABLE OF PRESENT VALUES OF AN ANNUITY
OF \$1 OR £1.

| No. of payments. | 3 per cent. | 4 per cent. | 5 per cent. | 6 per cent. |
|------------------|-------------|-------------|-------------|-------------|
| 1 | 0.97097 | 0.96154 | 0.95228 | 0.94340 |
| 2 | 1.91347 | 1.88039 | 1.86941 | 1.85339 |
| 3 | 2.82861 | 2.77519 | 2.75719 | 2.67301 |
| 4 | 3.71710 | 3.62909 | 3.54595 | 3.46510 |
| 5 | 4.57971 | 4.45182 | 4.32918 | 4.21236 |
| 6 | 5.41719 | 5.24214 | 5.07569 | 4.91732 |
| 7 | 6.23028 | 6.00205 | 5.78657 | 5.58238 |
| 8 | 7.01969 | 6.73274 | 6.46321 | 6.20979 |
| 9 | 7.78611 | 7.43533 | 7.10782 | 6.80169 |
| 10 | 8.53326 | 8.11089 | 7.72173 | 7.36069 |
| 11 | 9.25262 | 8.76058 | 8.30641 | 7.88687 |
| 12 | 9.95106 | 9.38567 | 8.86325 | 8.38384 |
| 13 | 10.63496 | 9.98565 | 9.39357 | 8.85298 |
| 14 | 11.29697 | 10.56312 | 9.89864 | 9.29498 |
| 15 | 11.93794 | 11.11849 | 10.37965 | 9.71225 |
| 16 | 12.56116 | 11.65229 | 10.83777 | 10.10589 |
| 17 | 13.16612 | 12.16567 | 11.27406 | 10.47726 |
| 18 | 13.75351 | 12.65910 | 11.68958 | 10.82760 |
| 19 | 14.32380 | 13.13394 | 12.08532 | 11.15811 |
| 20 | 14.87748 | 13.59032 | 12.46221 | 11.46992 |
| 21 | 15.41592 | 14.02916 | 12.82115 | 11.76407 |
| 22 | 15.93922 | 14.45111 | 13.16300 | 12.04158 |
| 23 | 16.44861 | 14.85648 | 13.48857 | 12.30338 |
| 24 | 16.93554 | 15.24606 | 13.79864 | 12.55036 |
| 25 | 17.41315 | 15.62208 | 14.09394 | 12.78335 |
| 26 | 17.87684 | 15.98277 | 14.37518 | 13.00316 |
| 27 | 18.32763 | 16.32958 | 14.64303 | 13.21053 |
| 28 | 18.76411 | 16.66306 | 14.89812 | 13.40616 |
| 29 | 19.18846 | 16.98371 | 15.14107 | 13.59072 |
| 30 | 19.60044 | 17.29203 | 15.37245 | 13.76483 |
| 31 | 20.00043 | 17.58849 | 15.59281 | 13.92968 |
| 32 | 20.38877 | 17.87355 | 15.80267 | 14.08404 |
| 33 | 20.76579 | 18.14764 | 16.00255 | 14.22923 |
| 34 | 21.13184 | 18.41119 | 16.19290 | 14.36814 |
| 35 | 21.48722 | 18.66461 | 16.37419 | 14.49824 |
| 36 | 21.83225 | 18.90828 | 16.54685 | 14.62069 |
| 37 | 22.16724 | 19.14258 | 16.71128 | 14.73678 |
| 38 | 22.49246 | 19.36786 | 16.86789 | 14.84692 |
| 39 | 22.80822 | 19.58448 | 17.01704 | 14.94997 |
| 40 | 23.11477 | 19.79277 | 17.15908 | 15.04630 |
| 41 | 23.41240 | 19.99305 | 17.29436 | 15.13501 |
| 42 | 23.70136 | 20.18562 | 17.42320 | 15.22454 |
| 43 | 23.98190 | 20.37079 | 17.54591 | 15.30617 |
| 44 | 24.25428 | 20.54844 | 17.66277 | 15.38318 |
| 45 | 24.51871 | 20.72004 | 17.77407 | 15.45583 |
| 46 | 24.77545 | 20.88465 | 17.88006 | 15.52437 |
| 47 | 25.02471 | 21.04293 | 17.98101 | 15.58903 |
| 48 | 25.26677 | 21.19513 | 18.07714 | 15.65002 |
| 49 | 25.50166 | 21.34166 | 18.16872 | 15.70757 |
| 50 | 25.72977 | 21.48297 | 18.25592 | 15.76186 |

APPLICATIONS.

56. To find the amount of an annuity forborne for any number of years at compound interest :

RULE.

$$A = \frac{a \{ (1+r)^t - 1 \}}{r}$$

INTERPRETATION.—From the amount raised to the power indicated by the number of payments subtract 1 and multiply the remainder by the annuity. Lastly: divide the sum thus obtained by the rate per unit and the quotient will be the required amount.

BY THE TABLE.—Find from the table the amount of \$1 for the given number of payments and at the given rate, multiply it by the given annuity and the quotient will be the amount.

EXAMPLE 1.—If a yearly rent of \$400 be forborne for 23 years, to what sum will it amount at 5 per cent. compound interest?

OPERATION.

Here $a = 400$, $t = 23$, $r = .05$.

$$\text{Then } A = \frac{a \{ (1+r)^t - 1 \}}{r} = \frac{400 \{ (1.05)^{23} - 1 \}}{.05} = \frac{400 \times 2.071475}{.05} = \frac{828.590}{.05} = \$16571.90. \text{ Ans.}$$

BY THE TABLE.—Amount of \$1 at the given rate and time = \$41.43047.

Then $\$41.43047 \times 400 = \16572.188 .

NOTE.—These two methods give results slightly different. This arises from the fact that the table shows only an approximation to the correct amount of the annuity for \$1; all the figures except the first five of its decimal being rejected.

57. To find the present value of an annuity at compound interest:—

RULE.

$$V = \frac{a}{r} \left\{ 1 - \frac{1}{(1+r)^t} \right\}$$

INTERPRETATION.—Divide 1 by that power of the amount of \$1 which is indicated by the number of payments and subtract the result from 1.

Multiply the remainder by the quotient arising from the division of the given annuity by the rate per unit and the result will be the required present value.

BY THE TABLE.—Find the present value of an annuity of \$1 for the given number of payments and at the given rate, and multiply this by the given annuity.

EXAMPLE 2.—What is the present value of an annuity at \$10, to continue 5 years, allowing 5 per cent. compound interest?

OPERATION.

Here $a = 40$, $t = 5$, and $r = .05$.

$$\text{Then } V = \frac{a}{r} \left\{ 1 - \frac{1}{(1+r)^t} \right\} = \frac{40}{.05} \times \left\{ 1 - \frac{1}{(1.05)^5} \right\} = \frac{4000}{5} \times (1-.7835) \\ = 800 \times .2165 = \$173.20. \text{ Ans.}$$

OR BY THE TABLE.—Present value of an annuity of \$1 for given rate and time = \$4.32948 and $\$4.32948 \times 40 = \173.179 . *Ans.*

58. To find the present worth of a perpetuity :—

RULE.

$$V = \frac{a}{r}.$$

INTERPRETATION.—Divide the annuity by the rate per unit and the quotient will be the value of the perpetuity.

EXAMPLE 3.—What is the present value of a freehold estate of \$75—allowing the purchaser 6 per cent. compound interest for his money?

OPERATION.

Here $a = 75$, and $r = .06$.

$$\text{Then } V = \frac{a}{r} = \frac{75}{.06} = \frac{7500}{6} = \$1250. \text{ Ans.}$$

59. To find the present worth of a perpetuity in reversion :—

RULE.

$$V = \frac{a}{r(1+r)^s}$$

INTERPRETATION.—Find that power of the amount of \$1 for one payment that is indicated by the number of payments that have to elapse before the annuity reverts, multiply this by the rate per unit and divide the given annuity by the product—the result will be the present value.

EXAMPLE 4.—What is the present value of the reversion of a perpetuity of \$79.20 per annum, to commence 7 years hence—allowing the buyer $4\frac{1}{2}$ per cent. for his money?

OPERATION.

Here $a = 79.20$, $s = 7$, and $r = .045$.

$$\text{Then } V = \frac{a}{r(1+r)^s} = \frac{79.20}{.045 \times (1-.045)^7} = \frac{79.20}{.045 \times 1.360862} = \frac{79.20}{.06123879} = \\ \$1293.297. \text{ Ans.}$$

60. With due attention to the foregoing interpretations and examples, the pupil will not experience any difficulty in applying the remaining formulas.

EXERCISE.

5. What is the annual rental of a freehold estate purchased for \$3000 when the rate of interest is at 4 per cent. ?
Ans. \$120.
6. If a perpetuity of \$563 can be purchased for \$11260 ready money, what is the rate of interest allowed ?
Ans. 5 per cent.
7. A freehold estate producing \$75 per annum, is mortgaged for the period of 14 years ; what is its present value, reckoning compound interest at 5 per cent. per annum ?
Ans. \$757·608.
8. Required the present value of a deferred annuity of \$90, to be entered upon at the expiration of 12 years, and then to be continued for seven years at 4 per cent. compound interest ?
Ans. \$337·39.
9. What is the present value of an estate whose rental is \$1500, allowing 5 per cent. compound interest ?
Ans. \$30000, or 20 years' purchase.
10. For how many years may an annuity of £22 be purchased for £308 12s. 10d., allowing compound interest at 4 per cent. ?
Ans. 21 years.
11. What is the present value of an annuity of \$154 for 19 years at 5 per cent. compound interest ?
Ans. \$1861·13.
12. What annuity, accumulating at $3\frac{1}{2}$ per cent. compound interest, will amount to £600 in 40 years ?
Ans. £6 13s. 11d.
13. In how many years will an annuity of \$8 per annum amount to \$187·315625 at 3 per cent. compound interest ?
Ans. 18 years.
14. What will an annuity of \$74 amount to in 30 years at 4 per cent. compound interest ?
Ans. \$4150·28.

 QUESTIONS TO BE ANSWERED BY THE PUPIL.

NOTE.—The numbers after the questions refer to the numbered articles of the section.

1. When are quantities said to be in arithmetical progression ? (1)
2. What are the extremes ? What the means ? (2)
3. What five quantities are to be considered in arithmetical progression ? (3)
4. How are these related to each other ? (3)
5. How many cases arise from these combinations ? (3)

6. Deduce the fundamental formulas for arithmetical progression. (4-7)
7. When are quantities said to be in geometrical progression? (15)
8. What five quantities are to be considered in geometrical progression? (16)
9. How are these related and how many cases arise from their combinations? (16)
10. Deduce the fundamental formulas for geometrical progression. (17-19)
11. What rule do you use when finding the sum of any infinite series when the ratio is less than 1? (25)
12. Prove this rule. (25)
13. How do we insert any number of arithmetical means between two given extremes? (26)
14. How do we insert any number of geometrical means between two extremes? (26)
15. What is position? (27)
16. Into what rules is position divided? (28)
17. When is a single position used? (29)
18. What class of questions require the use of double position? (30)
19. Give and prove the common rule for single position. (32)
20. Give and prove the common rule for double position. (34)
21. Deduce algebraically a complete set of rules for compound interest. (36)
22. What is an annuity? (41)
23. When is an annuity said to be in possession? (42)
24. What is a deferred annuity or an annuity in reversion? (43)
25. What is a contingent annuity? (45)
26. What is a perpetuity? (46)
27. When is an annuity said to be in arrears? (47)
28. What is the amount of an annuity? (48)
29. What is the present worth of an annuity? (49)
30. Deduce a set of rules for computing annuities at simple interest.
31. Illustrate the absurdity and injustice of computing the present value of annuities at simple interest. (51)
32. Deduce a set of rules for annuities at compound interest. (55)

EXAMINATION PROBLEMS.

FIRST SERIES.

1. Write down as one number seven trillions and ninety millions, and nineteen and four million two hundred thousand and six hundredths of trillionths.
2. Deduct 19 per cent. from \$7580 and divide the remainder among A, B, C, and D, so that A may have \$111·11 more than B; B \$50·90 more than C, and D one-third as much as A, B and C together.
3. A and B can perform a piece of work in 8 days, when the days are 12 hours long; A, by himself, can do it in 12 days, of 16 hours each. In how many days of 14 hours long will B do it?
4. Reduce £179 14s. 8½d. to dollars and cents, and divide the result by ·00000048.
5. What is the l. c. m. of 44, 18, 30, 77, 56 and 27?

6. In what time will any sum of money amount to 20 times itself at $5\frac{1}{4}$ per cent. simple interest?
7. Divide 7342163 *octenary* by 61351 *nonary*, and give the answer in the duodenary scale true to two places to the right of the separating point.
8. Multiply 43 lbs. 3 oz. 17 dwt. 11 grs. by $783\frac{1}{2}$.
9. Find the sum of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, and *ad infinitum*.
- $$\frac{2\frac{1}{2}}{3}$$
10. Divide $\frac{1}{2}$ of $\frac{2}{3}$ of 192 by $\frac{4\frac{1}{2}}{\frac{2}{3}}$
11. Extract the 17th root of 129140163.
12. There is a number consisting of two places of figures, which is equal to four times the sum of its digits, and if 18 be added to it, its digits will be inverted. What is the number?

SECOND SERIES.

13. Divide \$897.43 among A, B and C, so that B may have \$93.40 less than A, and \$69.18 more than C.
14. If 7 lbs. of wheat contain as much nutritive matter as 9 lbs. of rye, and 5 lbs. of rye as much as 8 lbs. of oats, and 13 lbs. of oats as much as 21 lbs. of buckwheat, and 27 lbs. of buckwheat as much as 20 lbs. of barley, and 24 lbs. of barley as much as 26 lbs. of peas, and 11 lbs. of peas as much as 35 lbs. of potatoes; how many pounds of potatoes contain as much nourishment as 16 lbs. of wheat?
15. Reduce $\frac{3}{5}$ of $4\frac{1}{2}$ of $7\frac{4}{5}$ of $\frac{9}{19\frac{1}{2}}$ of $\frac{5}{9}$ of 3 oz. 4 drs. 2 scr. 5 grains to the decimal of $\frac{6}{7}$ of $\frac{63}{2}$ of $\frac{3}{13}$ of $6\frac{1}{2}$ times 7 lbs. 3 oz., Apothecaries Weight.
16. From 623.42793 take 93.4267192; mark distinctly the resulting repetend.
17. If I own a vessel valued at \$7493 and wish to insure it at a premium of $4\frac{2}{3}$ per cent. so as to recover, in case of the destruction of the vessel, both the premium paid and the value of the vessel, for what sum must I insure?
18. If 18 men in 20 weeks, of 5 working days each, working 11 hours a day, dig 11 cellars, each 20 feet long, 16 feet wide

- and 5 feet deep ; how many men will be required to dig 24 cellars, each 22 feet square and 4 feet deep, in 36 weeks of 6 days each, working 9 hours per day ?
19. A certain number is divided by 9 and the quotient multiplied by 17 ; the product is then divided by 300 and 33 is added to the quotient ; the result is next divided by 3, and from this quotient 31 is subtracted, and the resulting difference divided by $12\frac{1}{2}$. Now $\frac{1}{2}$ of $\frac{2}{3}$ of $\frac{1}{4}$ of this last quotient is $2\frac{2}{5}$. Required the original number.
 20. What is the l. c. m. of 480, 768, 348, and 1176 ?
 21. What is the G. C. M. of 17598, 46090, and 171347 ?
 22. In a certain adventure A put in \$12000 for 4 months, then adding \$3000, he continued the whole 2 months longer ; B put in \$25000, and after 3 months took out \$10000, and continued the rest for 3 months longer ; C put in \$35000 for 2 months, then withdrawing $\frac{2}{7}$ of his stock, continued the remainder for 4 months longer ; they gained \$15000 : what was the share of each ?
 23. Three merchants traffic in company, and their stock is £400 ; the money of A continued in trade 5 months, that of B 6 months, and that of C 9 months ; and they gained £375, which they divide equally. What stock did each put in ?
 24. A fountain has 4 pipes, A, B, C, and D, and under it stands a cistern, which can be filled by A in 6, by B in 8, by C in 10, and by D in 12 hours ; the cistern has 4 pipes, E, F, G, and H ; and can be emptied by E in 6, by F in 5, by G in 4, and by H in 3 hours. Suppose the cistern is full of water, and that 8 pipes are all open, in what time will it be emptied ?

 THIRD SERIES.

25. Express 74938 and 17498679 in Roman Numerals.
26. 2310 loaves of bread are divided among charitable institutions in the following manner : as often as the first receives 4 the second receives 3, and as often as the first receives 6 the third gets 7 ; how many will each have ?
27. How much sugar at 4, 5, and 9 cents a pound, must be mixed with 72 pounds at 12 cents a pound, so that the mixture may be worth 8 cents a pound ?
28. What principal put out at interest will amount to \$4444.44 in 4 years, 4 months, 4 days, at 4.44 per cent. ?
29. For what sum must a ship valued at \$23470 be insured so as, in case of its destruction, to recover both the value of the vessel and the premium of $2\frac{1}{4}$ per cent. ?

30. What principal will amount to \$7493·47 in 8 years, allowing simple interest at 7 per cent. ?
31. I send to my agent in Manchester \$17460 and instruct him to deduct his commission at $3\frac{1}{2}$ per cent., and invest the balance in broadcloths at \$2·95 per yard. When I receive the goods I have to pay in addition \$1347·90 for carriage, \$479·40 for insurance, \$169·83 for storage, wharfage, and harbour dues, and an *ad valorem* duty at $2\frac{1}{2}$ per cent. on the invoice of goods. Required how many yards of cloth my agent ships to me and what I gain or lose per cent. on the whole transaction if I sell the goods for \$25000.
32. Transpose 134234 *quinary* into the *ternary*, *octenary*, and *duodenary* scales, and prove the results by reducing all four numbers to the *denary* scale.
33. What is the difference between $\frac{3}{4}$ of $4\frac{1}{2}$ of $\frac{9\frac{1}{2}}{\frac{1}{2}\frac{7}{8}}$ of $\frac{1}{18}$ of $\frac{7}{9}$ of £43 18s. 11½d., and $3\frac{2}{5}$ of $\frac{1}{17\frac{1}{2}}$ of ·56 of 1·75 of $6\frac{1}{2}$ times \$97·18?
34. Given the logarithm of $2 = 0\cdot301030$
 $3 = 0\cdot477121$
 $13 = 1\cdot113943$
 Find the logarithms of $\frac{1}{13}$, 19·5, 1125, 28·16, 65000, ·0005, 152·1, and 8·112.
35. Extract the cube root of 871tet·72 *duodenary* true to two places to the right of the separating point.
36. A person passed $\frac{1}{5}$ of his age in childhood, $\frac{1}{4}$ of it in youth, $\frac{1}{3}$ of it + 5 years in matrimony; he had then a son whom he survived 4 years, and who reached only $\frac{1}{2}$ the age of his father. At what age did this person die ?

FOURTH SERIES.

37. Divide 63 miles 3 fur. 7 per. 3 yds. 2 ft. 7 in. by 7 fur. 23 per. $3\frac{1}{2}$ yds.
38. Divide 6·3 by ·000000274.
39. If $\frac{7}{8}$ yards of cloth cost \$ $\frac{1}{2}$, how much will $6\frac{3}{4}$ yards cost?
40. Find the interest on \$4237·71 at $6\frac{1}{2}$ per cent. for 1·67 years.
41. In what time will \$674·30 amount to \$1000 at $8\frac{1}{2}$ per cent.?
42. What are the amount and compound interest of \$813·71 for 7 years at 4 per cent. half-yearly?
43. A owes B \$4300 to be paid as follows, viz.: \$300 down, \$700 at the end of 4 months, \$750 at the end of 7 months, \$850 at the end of 9 months, \$400 at the end of 13 months, and the balance at the end of 19 months. Required the equated time for the whole debt.

44. Deduct 23 per cent. from \$4200 and divide the remainder between A, B, C, D, and E, so that A may have \$17·10 more than B, C \$19·23 less than B, D \$42·11 less than C, and E half as much as A, B, C, and D together.
45. What principal put out at interest at 16 per cent. will amount to \$3786·80 in 11 years?
46. Find the value of
- $$\left\{ (3\frac{3}{4} - 2\frac{7}{16}) \times 4\frac{6}{10} \div \frac{2}{6} \text{ of } 14285\frac{7}{8} \right\} \div 8\frac{1}{2} \text{ times } (\frac{1}{2} + \frac{1}{7} + \frac{1}{8} - 2\frac{3}{376})$$
- $$\left\{ (.73 \times 12345 \div \frac{7}{10}) + \frac{2}{3} + 9\frac{2}{3} + 17\frac{1}{4} \right\} \div 27\cdot492207\bar{7}$$
47. Add together 312312302 and 2312132 *quaternary*; multiply the sum by twenty-three thousand and eleven times 4234 *quinary*; from the product subtract 555+444+333+222+111 *senary*; divide the remainder by 6542 *septenary*, and give the answer in the *octenary* scale.
48. What is the square of ·1 and also of ·i?

FIFTH SERIES.

49. Read the following numbers:
1000300050006000·000700080009.
7600290034007·000000067400209.
50. Find the l. c. m. of 2, 9, 16, 27, 48, and 81.
51. In what time will any sum of money amount to 7 times itself at 6 per cent. per annum compound interest?
52. How often will a coach wheel turn in going from Toronto to Brampton, a distance of 20 miles; the wheel being 14 ft. 10 in. in circumference?
53. How many divisors has the number 1749600?
54. Divide $\frac{3}{8}$ of $\frac{96}{8}$ by $\frac{\frac{1}{2} \text{ of } 7}{3\frac{1}{2}}$
- $$\frac{\frac{3}{8} \times \frac{96}{8}}{\frac{\frac{1}{2} \times 7}{3\frac{1}{2}}}$$
55. A can do a piece of work in 12 days, and A and B together can do it in 5 days; in what time can B alone do it?
56. What principal will amount to \$8899·77 in 11 years at 6 per cent. half yearly, compound interest?
57. Divide the number 10 into three such parts, that if the first be multiplied by 2, the second by 3, and the third by 4, the three products will be equal.
58. There are three fishermen, A, B, and C, who have each caught a certain number of fish; when A's fish and B's are put together, they make 110; when B's and C's are put together they make 130; and when A's and C's are put together they make 120. If the fish be divided equally among them, what will be each man's share; and how many fish did each of them catch?

59. What is the forty-seventh term and also the sum of the first 93 terms of the series 7, 11, 15, 19, &c. ?
60. In what time will any sum of money amount to 21 times itself at 7 per cent. compound interest ?

SIXTH SERIES.

61. Divide \$3700 among three persons, A, B, and C, so that B may have \$387 less than A and \$190.87 more than C.
62. What are all the divisors of 5716 ?
63. What is the value of
$$\frac{\{(177. - 10\frac{2}{3}) - (4 + \frac{1}{2} + 9 - \frac{1}{4})\} \div (8378 \div \frac{1}{2} \text{ of } 31)}{6322632 \times \frac{1}{2} \text{ of } \frac{3}{4} \div (\frac{1}{3} \text{ of } 4\frac{1}{2} \text{ of } \frac{1}{17} \text{ of } 85\frac{2}{3} \div 101)}$$
64. Divide \$7200 among 3 men, 4 women, and 17 children, giving each man twice as much as a woman, and each woman three times as much as a child. What is the share of each ?
65. How many divisors has the number 25400 ?
66. What is the difference between $\frac{3}{4}$ of $4\frac{1}{2}$ of $\frac{9\frac{3}{4}}{14}$ of $\frac{1}{6}$ of £3 16s. 11½d. and $\frac{3}{17}$ of $4\frac{3}{4}$ of $\frac{19\frac{1}{2}}{34}$ of $\frac{2\frac{1}{2}}{17}$ of $\frac{1}{2}$ of $\frac{1}{6}$ of .85 of $\frac{1}{42\frac{1}{2}}$ of \$1783 ?
67. Compare together the ratios 7:13, 9:16, 8:15 and 10:19 and point out which is the greatest, which the least and what the ratio compounded of these given ratios.
68. Divide 67.432 by 7.9036.
69. Reduce 9 per. 9 yds. 7 ft. 120 in. to the decimal of $\frac{1}{4}$ of $\frac{3}{8}$ of $\frac{7}{8}$ of 35 acres 2 roods.
70. Add together 17.0342, 27.06357, 98.123456, 829.6423 986.1234298, 9.876342, and 813.9864234567.
71. In the ruins of Persepolis there are two columns left standing upright. The one is 64 feet above the plain and the other 50. In a straight line between these stands a small statue the head of which is 97 feet from the top of the higher column and 86 feet from the top of the lower, the base of which is 76 feet from the base of the statue. Required the distance between the tops of the columns.
72. In a mixture of spirits and water, $\frac{1}{2}$ of the whole plus 25 gallons was spirits, but $\frac{1}{3}$ of the whole minus 5 gallons was water. How many gallons were there of each ?

SEVENTH SERIES.

73. Extract the square root of 401241·3424 in the *quinary* scale.
74. A father being asked by his son how old he was, replied, your age is now $\frac{1}{2}$ of mine; but 4 years ago it was only $\frac{1}{3}$ of what mine is now: what is the age of each?
75. Divide $\cdot 72347$ by $\cdot 0032$.
76. Extract the 11th root of 97294764·372.
77. Find two numbers, the difference of which is 30, and the relation between them as $7\frac{1}{2}$ is to $3\frac{1}{2}$?
78. What is the l. c. m. of 35, 16, 18, 28, 62, 63 and 40?
79. Sum the series $1+7+13+19+\&c.$, to 101 term.
80. What is the ratio compounded of 19:7, 11:56, 35:121, $11\frac{3}{4}:29$, 8:4 $\frac{3}{4}$ and $4\frac{1}{2}:3$.
81. Find two numbers whose sum and product are equal, neither of them being 2?

NOTE.—In this question take any number for the first of the two, as for example 7. Then $7 +$ some other number $= 7 \times$ that other number.

Assume for this second number any other, as 3.

Then $7 + 3 = 10 = 7 \times 3$, gives an error of -11 .

Assume some other for the second as 5.

Then $7 + 5 = 12 = 7 \times 5$ gives an error of -23

Then $\frac{23 \times 3 = 69}{11 \times 5 = 55}$ Whence second number $= \frac{14}{12} = 1\frac{1}{6}$.

82. Find the value of

$$\frac{\left\{ \left(9\frac{1}{2} + 4\frac{1}{2} + 3\frac{1}{2} - 16\frac{3}{4} \right) \times 54 \left\{ \div 1\frac{1}{2} \right\} \times 35 \text{ times } \cdot 14285\dot{7} \right.}{\left. \left\{ \cdot 97 \times \cdot 24378 \times \left(1\frac{1}{4} \times 4\frac{1}{4} \right) \right\} \times \left(4\frac{3}{4} - 2\frac{1}{4} \right) \right.}$$

83. The hour and minute hands of a watch are together at 12; when will they be together again?
84. Given the logarithm of 2 = 0·301030
 logarithm of 7 = 0·845098
 logarithm of 11 = 1·041303

Find the logarithms of 3850000, 3181·81, ·0000154, $\frac{1}{17}$
 1·571428 and 93·17.

EIGHTH SERIES.

85. Find the difference between the simple and compound interest of \$700 in 3 years at $4\frac{1}{2}$ per cent. per annum.
86. X, Y, and Z, form a company. X's stock is in trade 3 months, and he claims $\frac{1}{2}$ of the gain; Y's stock is 9 months in trade; and Z advanced \$3024 for 4 months, and claims half the profit. How much did X and Y contribute?

87. There is a fraction which multiplied by the cube of $1\frac{1}{2}$ and divided by the square root of $1\frac{3}{4}$, produces $\frac{2}{3}$, find it.
88. Find the cube root of 80677568161.
89. How much sugar, at 4d., 6d., and 8d. per lb. must there be in 112 lbs. of a mixture worth 7d. per lb.
90. Find three such numbers as that the first and $\frac{1}{2}$ the sum of the other two, the second and $\frac{1}{3}$ the sum of the other two, the third and $\frac{1}{4}$ the sum of the other two will make 34.

NOTE.—Assume 40 as the sum of the three numbers.

$$\begin{aligned} \text{Then } 1st+2nd+3rd &= 40 \text{ and } 1st+\frac{1}{2}(2nd+3rd) = 34 \therefore \frac{1}{2}(2nd+3rd) \\ &= 6 \text{ and } 2nd+3rd = 12. \end{aligned}$$

$$2nd+\frac{1}{3}(1st+3rd) = 34 \therefore \frac{2}{3}(1st+2nd) = 6 \text{ and } 1st+3rd = 9.$$

$$3rd+\frac{1}{4}(1st+2nd) = 34 \therefore \frac{3}{4}(1st+2nd) = 6 \text{ and } 1st+2nd = 8.$$

$$\text{Then adding these together, twice } (1st+2nd+3rd) = 29. \therefore 1st+2nd+3rd = 14\frac{1}{2} = \text{sum.}$$

But should equal 40—therefore error = $-25\frac{1}{2}$.

Similarly assume some other number and apply the rule and the true sum, 53 will be found, from which the numbers may be easily obtained.

91. Insert 4 arithmetical means between 1 and 40.
92. The sum of all the terms of a geometrical progression is 1860040, the last term is 1240029, and the ratio is 3. What is the first term?
93. If 6 apples and 7 pears cost 33 pence, and 10 apples and 8 pears 44 pence, what is the price of one apple and one pear?
94. Multiply $\frac{1}{2}$ of $\frac{1}{3}$ of $\frac{5}{9}$ of $\frac{28\frac{1}{2}}{6}$ by $\frac{2}{3}$ of $\frac{1}{2}$ of $\frac{1}{4}$.
95. From a sum of money, \$50 more than the half of it is first taken away; from the remainder, \$30 more than its fifth part; and again, from the second remainder, \$20 more than its fourth part. At last there remained only \$10. What was the original sum?
96. A gentleman hires a servant, and promises him, for the first year, only \$60 in wages, but for each following year \$4 more than the preceding. How much will the servant receive for the 17th year of his engagement, and how much for all 17 years together?

NINTH SERIES.

97. Write down as one number eleven trillions and eleven, and eleven tenths of billionths.
98. Reduce £749 16s. 5½d. sterling to dollars and cents.
99. What are the prime factors of 177408?
100. At what rate per cent. per annum will \$704 amount to \$11111.11 in 11 years at compound interest?

101. How many scholars are there in a school to which if 9 be added the number will be augmented by one-thirteenth.
102. Three different kinds of wine were mixed together in such a way that for every 3 gallons of one kind there were 4 of another, and 7 of a third: what quantity of each kind was there in a mixture of 292 gallons?
103. Divide £500 among four persons, so that when A has $\text{£}\frac{1}{2}$, B shall have $\text{£}\frac{1}{3}$, C $\frac{1}{4}$, and D $\frac{1}{5}$.
104. What is the present worth of an annuity of $\text{£}100$ to continue 23 years, at 6 per cent. compound interest?
105. Twenty-five workmen have agreed to labor 12 hours a day for 24 days, to pay an advance made to them of $\text{£}900$; but having each lost an hour per day, five of them engage to fulfil the agreement by working 12 days: how many hours per day must these labor?
106. A man has several sons, whose ages are in arithmetical progression; the age of the youngest is 5 years, the common difference of their ages is 6 years, and the sum of all their ages is 161. What is the age of the eldest?
107. If a man dig a small square cellar, which will measure 6 feet each way, in one day, how long will it take him to dig a similar one that shall measure 10 feet each way?
108. A servant agreed to live with his master for $\text{£}8$ a year, and a suit of clothes. But being turned out at the end of 7 months, he received only $\text{£}2$ 13s. 4d. and the suit of clothes: what was its value?

TENTH SERIES.

109. What number is that of which $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ added together, will make 48?
110. If an ox, whose girth is 6 feet, weighs 600 lbs., what is the weight of an ox whose girth is 8 feet?
111. Four women own a ball of butter, 5 inches in diameter. It is agreed that each shall take her share separately from the surface of the ball. How many inches of its diameter shall each take?
112. Divide $71213 \cdot 43$ by $12 \cdot 342$ in the *nonary* scale and extract the square root of the quotient true to three places to the right of the separating point.
113. Five merchants were in partnership for four years; the first put in $\text{\$}60$, then, 5 months after, $\text{\$}800$, and at length $\text{\$}1500$, 4 months before the end of the partnership; the second put in at first $\text{\$}600$, and 6 months after $\text{\$}1800$; the third put in $\text{\$}400$, and every six months after he

- added \$500; the fourth did not contribute till 8 months after the commencement of the partnership; he then put in \$900, and repeated this sum every 6 months; the fifth put in no capital, but kept the accounts, for which the others agreed to pay him \$1.25 a day. What is each one's share of the gain, which was \$2000?
114. In what time will any sum of money amount to 16 times itself at 5 per cent. per annum. 1st, at simple interest?
2nd, at compound interest?
115. Three persons purchased a house for \$9202; the first gave a certain sum; the second three times as much; and the third one and a half times as much as the two others together; what did each pay?
116. A piece of land of 163 acres was cleared by two companies of workmen; the first numbered 25 men and the second 22; how many acres did each company clear, and what did the clearing cost per acre, knowing that the first company received \$86 more than the second?
117. The greatest of two numbers is 15 and the sum of their squares is 346; what are the two numbers?
118. To what sum will \$1200 amount to in 10 years at $9\frac{1}{2}$ per cent?
119. If 496 men, in $5\frac{1}{2}$ days of 11 hours each, dig a trench of 7 degrees of hardness 465 feet long, $3\frac{3}{4}$ wide, $2\frac{1}{2}$ deep, in how many days of 9 hours long will 24 men dig a trench of 4 degrees of hardness $337\frac{1}{2}$ feet long, $5\frac{3}{4}$ wide, and $3\frac{1}{2}$ deep?
120. Four men, A, B, C, and D, took a prize of \$6213, which they are to divide in proportion to the following fractions: *if possible*, A, B, and C, are to have $\frac{1}{6}$; B, C, and D, $\frac{3}{7}$; A, C, and D, $\frac{1}{5}$; and A, B, and D, $\frac{1}{3}$ of the prize. What does each receive?

ELEVENTH SERIES.

121. Reduce $\cdot 7$, $\cdot 83$, $\cdot 727$, $\cdot 91325$ and $8\cdot 671347$ to their equivalent vulgar fractions.
122. Reduce $713\frac{1}{11}$ *undenary*, and $12123\frac{1}{100000}$ *quaternary* to equivalent expressions in the *denary* scale.
123. Add together $3\frac{1}{2}$ of $2\frac{1}{2}$ of $7\frac{1}{2}$ of a £, $9\frac{1}{2}$ of $3\frac{1}{2}$ of a shilling, and $8\frac{1}{2}$ of $4\frac{1}{2}$ of a penny, and divide the sum by $\frac{1}{2}$ of $\frac{1}{4}$ of $\frac{1}{2}$ of $3\frac{1}{2}$ d.
24. If 24 pioneers, in $2\frac{1}{2}$ days of $12\frac{1}{2}$ hours long, can dig a trench 139.75 yds. long, $4\frac{1}{2}$ yds. wide, and $2\frac{1}{2}$ yds. deep, what length of trench will 90 pioneers dig in $4\frac{1}{2}$ days of $9\frac{1}{2}$ hours long, the trench being $4\frac{1}{2}$ yds. wide and $3\frac{1}{2}$ yds. deep?

125. A person, by disposing of goods for \$182, loses at the rate of 9 per cent.; what ought they to have been sold for to realize a profit of 7 per cent.?
126. In what time will any sum of money amount to $11\frac{1}{2}$ times itself at 6 per cent. per annum.
 1st At simple interest?
 2nd At compound interest?
127. It is desired to cut off an acre of land from a field $15\frac{1}{2}$ perches in breadth; what length must be taken?
128. Express a degree ($69\frac{1}{2}$ miles) in metres, when 32 metres are equal to 35 yds.
129. Find 7 geometrical means between 3 and 19683.
130. Sum the infinite series $7 + 1\frac{1}{3} + \frac{1}{27}$, &c.
131. Four men bought a grindstone of 60 inches diameter. Now how much of the diameter must be ground off by each man, one grinding his part first, then another, and so on, that each may have an equal share of the stone, no allowance being made for the axle?
132. Divide 100 guineas into an equal number of guineas, half-guineas, crowns, half-crowns, shillings, and sixpences, and reduce the remainder to a fraction of a pound.

TWELFTH SERIES.

133. The owner of $\frac{1}{7}$ of a ship sold $\frac{1}{4}$ of $\frac{2}{3}$ of his share for \$12 $\frac{3}{4}$; what would $\frac{2\frac{1}{2}}{4\frac{1}{4}}$ of $\frac{2}{3}$ cost at the same rate?
134. At what rate per cent. per annum will \$700.90 amount to \$1679.40 in 5 years, compound interest being allowed?
135. A person paid a tax of 10 per cent. on his income; what must his income have been, when, after he had paid the tax, there was \$1250 remaining?
136. The sum of £3 13s. 6d. is to be divided among 21 men, 21 women, and 21 children, so that a woman may have as much as two children, and a man as much as a woman and a child: what will each man, woman and child receive?
137. Distribute \$200 among A, B, C, and D, so that B may receive as much as A; C as much as A and B together, and D as much as A, B, and C, together.
138. Find the difference between $\sqrt{\frac{2}{3}}$ and $\sqrt[3]{\frac{2}{3}}$.
139. Reduce $3\frac{2}{3}$, $17\frac{5}{2} + \frac{1}{5} + 144\frac{1}{4}$, $2\frac{3}{8} - \frac{1}{11}$, $\frac{2}{3}$ of $\frac{1}{4} \times \frac{1}{5}$ of $\frac{1}{4}$ of $\frac{1}{2}$, and $6347 \div 2\frac{3}{4}$, to their simplest forms.
140. Find the cube root of 884736, and the fourth root of 95951 $\frac{1}{8}\frac{1}{2}$.

141. A general levied a contribution of \$520 on four villages, containing 250, 300, 400 and 500 inhabitants respectively; what must they each pay?
142. A person had a salary of \$520 a year, and let it remain unpaid for 17 years. How much had he to receive at the end of that time, allowing 6 per cent. per annum compound interest, payable half-yearly?
143. Insert four arithmetical means between 2 and 79; also find the 9th term and the sum of the first 207 term of the series 3, 7, 11, 15, &c.
144. A, B, and C, start at the same time, from the same point, and in the same direction, round an island 73 miles in circumference; A goes at the rate of 6, B at the rate of 10, and C at the rate of 16 miles per day. In what time will they be all together again?

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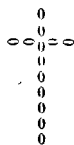
ARITHMETICAL RECREATIONS.

1. If the third of 6 be 3 what must the fourth of 20 be?
2. If the half of 5 be 7 what part of 9 will be 11?
3. Place four *nines* so that their sum shall be 100.
4. What part of three pence is the third of two pence?
5. If a herring and a half cost $1\frac{1}{2}$ d. how much will eleven herrings cost?
6. If 12 apples are worth 21 pears and 3 pears cost a cent, what will be the price of 100 apples?
7. Find a number such that 5 shall be the three-sevenths of it?
8. A hundred hurdles are so placed as to inclose 200 sheep, and with two hurdles more the field may be made to hold 400; how is this to be done?
9. A gentleman who owned four hundred acres of land in the form of a square, desired to keep 100 acres, also in the form of a square in one corner, and divide the remainder *a b c d e f*, equally among his four sons, so that each son, should have his lot of the same shape as his brothers. How may this be done?
10. Place four *threes* so as to make 34.
11. Write down 13 in such a way that rubbing half of it out 8 shall remain.
12. Two thirsty persons cast away on a desert island, find an 8-gallon cask of water. They wish to divide it equally between them but have no other measures than the eight-gallon cask, a five-gallon cask and a three-gallon cask. How can they divide it?
13. How must a board 16 inches long and 9 inches wide be cut into two such parts, that when they are joined together they may form a square?
14. Place the 9 digits in the accompanying figure, one digit to each cell, in such a way that when added vertically, horizontally or diagonally, the sum shall always be the same,

15. Three persons bought a quantity of sugar weighing 51 lbs., and wish to part it equally between them. They have no weights but a 4 lbs. weight and a 7 lbs. weight. How can they divide it?
16. Suppose 26 hurdles can be placed in a rectangular form so as to inclose 40 square yards of ground; how can they be placed when two of them are taken away, so as to inclose 120 square yards?
17. A person has a fox, a goose and a peck of oats to carry over a river, but on account of the smallness of the boat he can only carry over one at a time. How can this be done so as not to leave the fox with the goose, nor the goose with the oats?
18. The sleeping apartments of a certain boarding school consisted of nine rooms of which the centre one was occupied by a blind porter and the rest by the boys. The good porter to assure himself that all were in, visited all the rooms and finding 3 boys in each which made 9 in each row, retired to rest. Four boys however went out, and the porter returning to count them, still found nine in each row, and therefore retired as before. The four boys then came back, bringing each another boy with him, and the porter upon paying them another visit, counted them as before, and entertained no suspicion of what had taken place. After this four more strange boys were introduced and the porter still found the number apparently the same, i. e., nine in each row. Again four more were introduced, making the number of strange boys twelve, and still the porter was satisfied. Finally the twelve strange boys went away, taking with them six of the school boys, and the porter again counting them, retired in the full persuasion that no one had gone out or come in. How was all this possible?
19. Three jealous husbands and their wives having to cross a river, find a boat without its owner, which can only carry two persons at a time; in what manner then, can these six persons transport themselves over by pairs, so that none of the women shall be left in company with any of the men except when her husband is present?
20. Place the first 25 numbers 1, 2, 3, 4, 5, &c., in the cells of the accompanying figure, so that the columns added in any order, i. e., vertically, horizontally or diagonally, may amount to the same sum.

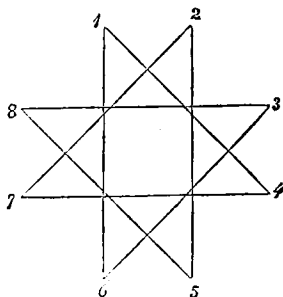


21. What is the difference between half-a-dozen dozen and six dozen dozen?
22. If a cross be made of 13 counters as in the margin, *nine* may be reckoned in three ways, i. e., by counting from the bottom up to the top of the perpendicular line; from the bottom up to the cross and then to the right; or from the bottom up to the cross and then to the left. Now take away two of the counters and with the others form a cross which shall possess the same property of counting *nine* when thus reckoned.
23. Seven out of 21 bottles being full of wine, 7 half full and 7 empty—it is required to distribute them among 3 persons, so that each may have the same quantity of wine and the same number of bottles.
24. Two travellers, one of whom had with him 5 bottles of wine and the other 3, were joined by a third person, who, after the wine was drunk left 8 shillings for his just share of it; how is this to be divided between the other two?
25. A person having by accident broken a basket of eggs, offered to pay for them on the spot if the owner could tell how many he had; to which he replied that he only knew there was between 50 and 100, and that when he counted them by 2's and 3's at a time none remained; but when he counted them by 5 at a time, there were 3 remaining; how many eggs had he?



26. It is required to find 4 such weights that they may weigh any number of pounds from 1 to 40.

27. In the accompanying figure it is required to fill seven out of the eight points with counters in the following manner, i. e., the counter has to start from an unoccupied point, pass along the line and be deposited at the other extremity. Thus, in commencing, the counter may start from any point since all are unoccupied, starting from 1 the counter may be carried either to 6 or to 4 and there deposited, suppose it be deposited at 6, then the next counter may start from any point except 6 and so on.



28. A brazen lion, placed in the middle of a reservoir, throws out water from its mouth, its eyes and its right foot. When the water flows from its mouth alone, it fills the reservoir in 6 hours; from the right eye it fills it in 2 days; from the left eye in 3 days, and from the foot in 4 days. In what time will the basin be filled by the water flowing from all these apertures at once?
29. Desire a person to think of any three numbers, each less than 10, and then tell him the numbers thought of.
30. Three men, Jones, Brown, and Smith, with their sons Harry, Tom and Ned, had each a piece of land in the form of a square. Jones' piece was 23 rods longer on each side than Tom's, and Brown's piece was 11 rods longer on each side than Harry's. Each man possessed 63 square rods of land more than his son. Which of the persons were father and son respectively.
31. A sea-captain, on a voyage had a crew of 30 men, half of whom were blacks. Being becalmed on the passage for a long time, their provisions began to fail, and the captain became satisfied that, unless the number of men were greatly diminished, all would perish of hunger before they could reach any friendly port. He therefore proposed to the sailors that they should stand in a row on deck, and that every ninth man should be thrown over-board, until one-half of the crew were thus destroyed. To this they all agreed. How should they stand so as to save the whites?
32. Direct a person to multiply together two numbers, one of which you select, and, unseen by you, to rub out one of the digits of the product—it is required to tell, upon his reading the remaining digits of the product, what figure was rubbed out.
33. It is required to write down beforehand the answer to a question in addition of a given number of lines, you writing the *second, fourth, sixth, &c.*, addends, and some other person the intermediate ones.

MATHEMATICAL TABLES.

LOGARITHMS OF NUMBERS FROM 1 TO 10,000, WITH
DIFFERENCES AND PROPORTIONAL PARTS.

| Numbers from 1 to 100. | | | | | | | | | |
|------------------------|----------|-----|----------|-----|----------|-----|----------|-----|----------|
| No. | Log. | No. | Log. | No. | Log. | No. | Log. | No. | Log. |
| 1 | 0.000000 | 21 | 1.322219 | 41 | 1.612784 | 61 | 1.785330 | 81 | 1.908485 |
| 2 | 0.301036 | 22 | 1.342423 | 42 | 1.623249 | 62 | 1.792392 | 82 | 1.913814 |
| 3 | 0.477121 | 23 | 1.361728 | 43 | 1.633468 | 63 | 1.799341 | 83 | 1.919078 |
| 4 | 0.602060 | 24 | 1.380211 | 44 | 1.643453 | 64 | 1.806180 | 84 | 1.924279 |
| 5 | 0.698970 | 25 | 1.397940 | 45 | 1.653213 | 65 | 1.812903 | 85 | 1.929419 |
| 6 | 0.778151 | 26 | 1.414973 | 46 | 1.662758 | 66 | 1.819544 | 86 | 1.934498 |
| 7 | 0.845098 | 27 | 1.431361 | 47 | 1.672098 | 67 | 1.826075 | 87 | 1.939519 |
| 8 | 0.903090 | 28 | 1.447153 | 48 | 1.681241 | 68 | 1.832509 | 88 | 1.944483 |
| 9 | 0.954243 | 29 | 1.462398 | 49 | 1.690196 | 69 | 1.838849 | 89 | 1.949390 |
| 10 | 1.000000 | 30 | 1.477121 | 50 | 1.698970 | 70 | 1.845098 | 90 | 1.954243 |
| 11 | 1.041303 | 31 | 1.491362 | 51 | 1.707570 | 71 | 1.851258 | 91 | 1.959041 |
| 12 | 1.079181 | 32 | 1.505150 | 52 | 1.716003 | 72 | 1.857332 | 92 | 1.963788 |
| 13 | 1.113943 | 33 | 1.518514 | 53 | 1.724276 | 73 | 1.863323 | 93 | 1.968483 |
| 14 | 1.146128 | 34 | 1.531479 | 54 | 1.732391 | 74 | 1.869232 | 94 | 1.973128 |
| 15 | 1.176091 | 35 | 1.544068 | 55 | 1.740363 | 75 | 1.875061 | 95 | 1.977724 |
| 16 | 1.204120 | 36 | 1.556303 | 56 | 1.748188 | 76 | 1.880814 | 96 | 1.982271 |
| 17 | 1.230449 | 37 | 1.568202 | 57 | 1.755875 | 77 | 1.886491 | 97 | 1.986772 |
| 18 | 1.255273 | 38 | 1.579784 | 58 | 1.763428 | 78 | 1.892095 | 98 | 1.991226 |
| 19 | 1.278754 | 39 | 1.591065 | 59 | 1.770852 | 79 | 1.897627 | 99 | 1.995635 |
| 20 | 1.301030 | 40 | 1.602060 | 60 | 1.778151 | 80 | 1.903090 | 100 | 2.000000 |

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| | 100 | 00000 | 000134 | 000868 | 001301 | 001734 | 002166 | 002598 | 003029 | 003461 | 003891 | 432 |
| 41 | 1 | 4321 | 4751 | 5181 | 5609 | 6034 | 6466 | 6894 | 7321 | 7748 | 8174 | 428 |
| 83 | 2 | 8600 | 9026 | 9451 | 9876 | 10300 | 10724 | 11147 | 11570 | 11993 | 12415 | 424 |
| 124 | 3 | 01287 | 012329 | 013850 | 014400 | 4521 | 4940 | 5360 | 5779 | 6197 | 6616 | 420 |
| 166 | 4 | 7033 | 7451 | 7868 | 8284 | 8700 | 9116 | 9532 | 9947 | 10361 | 10775 | 416 |
| 207 | 5 | 02189 | 021003 | 022016 | 022428 | 022841 | 023252 | 023664 | 024075 | 4496 | 4896 | 412 |
| 248 | 6 | 5396 | 5715 | 6125 | 6533 | 6942 | 7350 | 7757 | 8164 | 8571 | 8978 | 408 |
| 289 | 7 | 9541 | 9759 | 030105 | 030900 | 031694 | 032488 | 033282 | 034076 | 034870 | 035664 | 404 |
| 331 | 8 | 033124 | 033826 | 4227 | 4628 | 5029 | 5430 | 5830 | 6230 | 6629 | 7028 | 400 |
| 373 | 9 | 7426 | 7825 | 8223 | 8620 | 9017 | 9414 | 9811 | 1040207 | 104602 | 104998 | 397 |
| | 110 | 041203 | 041787 | 042182 | 042576 | 042969 | 043362 | 043755 | 044148 | 044540 | 044932 | 393 |
| 38 | 1 | 5323 | 5714 | 6105 | 6495 | 6885 | 7275 | 7664 | 8053 | 8442 | 8830 | 390 |
| 76 | 2 | 9218 | 9606 | 9993 | 105030 | 050766 | 051153 | 051538 | 051924 | 052309 | 052694 | 386 |
| 113 | 3 | 053978 | 053463 | 053846 | 4230 | 4613 | 4996 | 5378 | 5760 | 6142 | 6524 | 383 |
| 151 | 4 | 6095 | 7286 | 7666 | 8046 | 8426 | 8805 | 9185 | 9563 | 9942 | 10320 | 379 |
| 189 | 5 | 060498 | 061075 | 061452 | 061829 | 062206 | 062582 | 062958 | 063333 | 063709 | 4083 | 376 |
| 227 | 6 | 4448 | 4832 | 5206 | 5580 | 5953 | 6326 | 6699 | 7071 | 7443 | 7815 | 373 |
| 265 | 7 | 8106 | 8557 | 8928 | 9298 | 9668 | 10035 | 10400 | 10765 | 11129 | 11493 | 370 |
| 302 | 8 | 071882 | 072259 | 072637 | 072985 | 073352 | 3718 | 4085 | 4451 | 4816 | 5182 | 366 |
| 340 | 9 | 5547 | 5912 | 6276 | 6640 | 7004 | 7368 | 7731 | 8094 | 8457 | 8819 | 363 |
| | 120 | 079181 | 079543 | 079904 | 080266 | 080626 | 080987 | 081347 | 081707 | 082067 | 082426 | 360 |
| 35 | 1 | 082785 | 083144 | 083503 | 3-61 | 4219 | 4576 | 4934 | 5291 | 5647 | 6004 | 357 |
| 70 | 2 | 6360 | 6716 | 7071 | 7426 | 7781 | 8136 | 8490 | 8845 | 9198 | 9552 | 353 |
| 104 | 3 | 9905 | 090258 | 090611 | 090963 | 091315 | 091667 | 092018 | 092370 | 092721 | 093071 | 352 |
| 139 | 4 | 093422 | 3772 | 4122 | 4471 | 4820 | 5169 | 5518 | 5866 | 6215 | 6562 | 349 |
| 174 | 5 | 6910 | 7257 | 7604 | 7951 | 8298 | 8644 | 8990 | 9335 | 9681 | 10026 | 346 |
| 210 | 6 | 106571 | 107157 | 107743 | 108328 | 108914 | 109500 | 110084 | 110670 | 111255 | 111840 | 343 |
| 244 | 7 | 3204 | 4146 | 4887 | 4828 | 5169 | 5510 | 5851 | 6191 | 6531 | 6871 | 341 |
| 278 | 8 | 7210 | 7549 | 7888 | 8227 | 8565 | 8903 | 9241 | 9579 | 9916 | 10254 | 338 |
| 313 | 9 | 110590 | 110926 | 111263 | 111599 | 111934 | 112270 | 112605 | 112940 | 113275 | 3699 | 335 |
| | 130 | 113943 | 114277 | 114611 | 114944 | 115278 | 115611 | 115943 | 116276 | 116608 | 116940 | 333 |
| 32 | 1 | 7271 | 7603 | 7934 | 8265 | 8595 | 8926 | 9256 | 9586 | 9915 | 10245 | 330 |
| 61 | 2 | 120574 | 120903 | 121231 | 121560 | 121888 | 122216 | 122544 | 122871 | 123200 | 123528 | 328 |
| 97 | 3 | 3852 | 4178 | 4504 | 4830 | 5156 | 5481 | 5806 | 6131 | 6456 | 6781 | 325 |
| 129 | 4 | 7195 | 7429 | 7753 | 8076 | 8399 | 8722 | 9045 | 9368 | 9690 | 130012 | 323 |
| 161 | 5 | 130434 | 130655 | 130977 | 131298 | 131619 | 131939 | 132260 | 132580 | 132900 | 3219 | 321 |
| 193 | 6 | 3539 | 3858 | 4177 | 4496 | 4814 | 5133 | 5451 | 5769 | 6086 | 6403 | 318 |
| 225 | 7 | 6721 | 7037 | 7354 | 7671 | 7987 | 8303 | 8618 | 8934 | 9249 | 9564 | 316 |
| 258 | 8 | 9879 | 100194 | 100508 | 100822 | 101136 | 101450 | 101763 | 102076 | 102389 | 102702 | 314 |
| 290 | 9 | 143015 | 3327 | 3639 | 3951 | 4263 | 4574 | 4885 | 5196 | 5507 | 5818 | 311 |
| | 140 | 146128 | 146438 | 146748 | 147058 | 147367 | 147676 | 147985 | 148294 | 148603 | 148911 | 309 |
| 30 | 1 | 9219 | 9527 | 9836 | 150142 | 150449 | 150756 | 151063 | 151370 | 151676 | 151982 | 307 |
| 60 | 2 | 152288 | 152594 | 152900 | 3205 | 3510 | 3815 | 4120 | 4424 | 4728 | 5032 | 305 |
| 90 | 3 | 5326 | 5640 | 5943 | 6246 | 6549 | 6852 | 7154 | 7457 | 7759 | 8061 | 303 |
| 120 | 4 | 8362 | 8664 | 8965 | 9266 | 9567 | 9868 | 10168 | 10469 | 10769 | 11068 | 301 |
| 150 | 5 | 161958 | 161657 | 161956 | 162256 | 162554 | 162853 | 3161 | 3460 | 3758 | 4055 | 299 |
| 180 | 6 | 4535 | 4859 | 4947 | 5244 | 5541 | 5838 | 6134 | 6430 | 6726 | 7022 | 297 |
| 210 | 7 | 7317 | 7613 | 7908 | 8203 | 8497 | 8792 | 9086 | 9380 | 9674 | 9968 | 295 |
| 240 | 8 | 170262 | 170555 | 170848 | 171141 | 171434 | 171726 | 172019 | 172311 | 172603 | 172895 | 293 |
| 270 | 9 | 3185 | 3478 | 3769 | 4060 | 4351 | 4641 | 4932 | 5222 | 5512 | 5802 | 291 |
| | 150 | 176091 | 176381 | 176670 | 176959 | 177248 | 177536 | 177825 | 178113 | 178401 | 178689 | 289 |
| 28 | 1 | 8977 | 9264 | 9552 | 9839 | 180126 | 180413 | 180699 | 180986 | 181272 | 181558 | 287 |
| 56 | 2 | 181844 | 182129 | 182415 | 182700 | 2085 | 3270 | 3555 | 3839 | 4123 | 4407 | 285 |
| 84 | 3 | 4691 | 4975 | 5259 | 5542 | 5825 | 6108 | 6391 | 6674 | 6956 | 7239 | 283 |
| 112 | 4 | 7521 | 7803 | 8084 | 8366 | 8647 | 8928 | 9209 | 9490 | 9771 | 10051 | 281 |
| 140 | 5 | 190332 | 190612 | 190892 | 191171 | 191451 | 191730 | 192010 | 192289 | 192567 | 2846 | 279 |
| 168 | 6 | 3125 | 3403 | 3681 | 3959 | 4237 | 4514 | 4792 | 5069 | 5346 | 5623 | 278 |
| 196 | 7 | 5900 | 6176 | 6453 | 6729 | 7005 | 7281 | 7556 | 7832 | 8107 | 8382 | 276 |
| 224 | 8 | 8657 | 8932 | 9206 | 9481 | 9755 | 200229 | 200503 | 200777 | 201051 | 201324 | 274 |
| 252 | 9 | 201397 | 201670 | 201943 | 202216 | 202488 | 2761 | 3033 | 3305 | 3577 | 3848 | 272 |

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| 53 | 1 | 6826 | 7096 | 7365 | 7634 | 7904 | 8173 | 8441 | 8710 | 8979 | 9247 | 269 |
| 79 | 2 | 9515 | 9783 | 210051 | 210319 | 210586 | 210853 | 211121 | 211388 | 211654 | 211921 | 267 |
| 105 | 3 | 212188 | 212454 | 212721 | 212987 | 213252 | 213518 | 213783 | 214049 | 214314 | 214579 | 266 |
| 132 | 4 | 4844 | 5109 | 5373 | 5638 | 5902 | 6166 | 6430 | 6694 | 6957 | 7221 | 264 |
| 158 | 5 | 7484 | 7747 | 8010 | 8273 | 8536 | 8798 | 9060 | 9323 | 9585 | 9846 | 262 |
| 184 | 6 | 220108 | 220376 | 220643 | 220912 | 221181 | 221451 | 221721 | 221991 | 222261 | 222531 | 261 |
| 210 | 7 | 2716 | 2976 | 3236 | 3496 | 3755 | 4015 | 4274 | 4533 | 4792 | 5051 | 259 |
| 237 | 8 | 5309 | 5568 | 5825 | 6081 | 6242 | 6600 | 6858 | 7115 | 7372 | 7630 | 258 |
| | 9 | 7887 | 8144 | 8400 | 8657 | 8913 | 9170 | 9426 | 9682 | 9938 | 230193 | 256 |
| 25 | 170 | 230449 | 230704 | 230960 | 231215 | 231470 | 231724 | 231979 | 232234 | 232488 | 232742 | 254 |
| 50 | 1 | 2996 | 3250 | 3504 | 3757 | 4011 | 4264 | 4517 | 4770 | 5023 | 5275 | 253 |
| 74 | 2 | 5828 | 6081 | 6333 | 6585 | 6837 | 7089 | 7341 | 7592 | 7844 | 7795 | 252 |
| 99 | 3 | 8046 | 8297 | 8548 | 8799 | 9049 | 9299 | 9550 | 9800 | 20050 | 240300 | 250 |
| 124 | 4 | 240549 | 240799 | 241048 | 241297 | 241546 | 241795 | 242044 | 242293 | 242541 | 242790 | 249 |
| 149 | 5 | 3038 | 3286 | 3534 | 3782 | 4030 | 4277 | 4525 | 4772 | 5019 | 5266 | 248 |
| 174 | 6 | 5513 | 5759 | 6006 | 6252 | 6499 | 6745 | 6991 | 7237 | 7482 | 7728 | 246 |
| 198 | 7 | 7973 | 8219 | 8464 | 8709 | 8954 | 9198 | 9443 | 9687 | 9932 | 2250176 | 245 |
| 223 | 8 | 250240 | 250664 | 250905 | 251151 | 251395 | 251638 | 251881 | 252125 | 252368 | 2610 | 243 |
| | 9 | 2853 | 3096 | 3338 | 3580 | 3822 | 4064 | 4306 | 4548 | 4790 | 5031 | 242 |
| 24 | 180 | 255273 | 255514 | 255755 | 255996 | 256237 | 256477 | 256718 | 256958 | 257198 | 257439 | 241 |
| 47 | 1 | 7679 | 7918 | 8158 | 8398 | 8637 | 8877 | 9116 | 9355 | 9594 | 9833 | 239 |
| 71 | 2 | 260071 | 260310 | 260548 | 260787 | 261025 | 261263 | 261501 | 261739 | 261977 | 262214 | 238 |
| 94 | 3 | 2451 | 2698 | 2945 | 3192 | 3439 | 3686 | 3933 | 4180 | 4426 | 4673 | 237 |
| 118 | 4 | 4818 | 5064 | 5310 | 5557 | 5761 | 5966 | 6212 | 6467 | 6722 | 6977 | 235 |
| 141 | 5 | 7172 | 7406 | 7641 | 7875 | 8110 | 8344 | 8578 | 8812 | 9046 | 9279 | 234 |
| 165 | 6 | 9513 | 9746 | 9980 | 270213 | 270446 | 270679 | 270912 | 271144 | 271377 | 271609 | 233 |
| 188 | 7 | 271842 | 272074 | 272306 | 2538 | 2770 | 3001 | 3233 | 3464 | 3696 | 3927 | 232 |
| 212 | 8 | 4158 | 4389 | 4620 | 4850 | 5081 | 5311 | 5542 | 5772 | 6002 | 6232 | 230 |
| | 9 | 6462 | 6692 | 6921 | 7151 | 7380 | 7609 | 7838 | 8067 | 8296 | 8525 | 229 |
| 22 | 190 | 278784 | 278982 | 279211 | 279439 | 279667 | 279895 | 280123 | 280351 | 280578 | 280806 | 228 |
| 45 | 1 | 281033 | 281261 | 281488 | 281715 | 281942 | 282169 | 282396 | 282622 | 282849 | 283075 | 227 |
| 67 | 2 | 3301 | 3527 | 3753 | 3979 | 4205 | 4431 | 4656 | 4882 | 5107 | 5332 | 226 |
| 89 | 3 | 5557 | 5782 | 6007 | 6232 | 6456 | 6681 | 6905 | 7130 | 7354 | 7577 | 225 |
| 112 | 4 | 7802 | 8026 | 8249 | 8473 | 8696 | 8920 | 9143 | 9366 | 9589 | 9812 | 223 |
| 134 | 5 | 290035 | 290257 | 290480 | 290702 | 290925 | 291147 | 291369 | 291591 | 291813 | 292034 | 222 |
| 156 | 6 | 2256 | 2478 | 2699 | 2920 | 3141 | 3363 | 3584 | 3804 | 4025 | 4246 | 221 |
| 178 | 7 | 4466 | 4687 | 4907 | 5127 | 5347 | 5567 | 5787 | 6007 | 6226 | 6446 | 220 |
| 201 | 8 | 6665 | 6884 | 7104 | 7323 | 7542 | 7761 | 7979 | 8198 | 8416 | 8635 | 219 |
| | 9 | 8853 | 9071 | 9289 | 9507 | 9725 | 9943 | 300378 | 300378 | 300378 | 300378 | 218 |
| 21 | 200 | 301030 | 301247 | 301464 | 301681 | 301898 | 302114 | 302331 | 302547 | 302764 | 302980 | 217 |
| 42 | 1 | 3196 | 3412 | 3628 | 3844 | 4059 | 4275 | 4491 | 4706 | 4921 | 5136 | 216 |
| 64 | 2 | 5351 | 5566 | 5781 | 5996 | 6211 | 6425 | 6639 | 6854 | 7068 | 7282 | 215 |
| 85 | 3 | 7496 | 7710 | 7924 | 8137 | 8351 | 8564 | 8778 | 8991 | 9204 | 9417 | 213 |
| 106 | 4 | 9630 | 9843 | 310056 | 310268 | 310481 | 310693 | 310906 | 311118 | 311330 | 311542 | 212 |
| 127 | 5 | 311754 | 311966 | 2177 | 2389 | 2600 | 2812 | 3023 | 3234 | 3445 | 3656 | 211 |
| 148 | 6 | 3867 | 4078 | 4289 | 4499 | 4710 | 4920 | 5130 | 5340 | 5551 | 5761 | 210 |
| 170 | 7 | 5970 | 6180 | 6390 | 6599 | 6809 | 7018 | 7227 | 7436 | 7646 | 7854 | 209 |
| 191 | 8 | 8063 | 8272 | 8481 | 8689 | 8898 | 9106 | 9314 | 9522 | 9730 | 9938 | 208 |
| | 9 | 320162 | 320351 | 320562 | 320769 | 320977 | 321184 | 321391 | 321598 | 321805 | 322012 | 207 |
| 20 | 210 | 322219 | 322426 | 322633 | 322839 | 323046 | 323252 | 323458 | 323665 | 323871 | 324077 | 206 |
| 40 | 1 | 4282 | 4488 | 4694 | 4899 | 5105 | 5310 | 5516 | 5721 | 5926 | 6131 | 205 |
| 61 | 2 | 6336 | 6541 | 6745 | 6950 | 7155 | 7359 | 7563 | 7767 | 7972 | 8176 | 204 |
| 81 | 3 | 8380 | 8583 | 8787 | 8991 | 9194 | 9398 | 9601 | 9805 | 300008 | 320211 | 203 |
| 101 | 4 | 330414 | 330617 | 330819 | 331022 | 331225 | 331427 | 331630 | 331832 | 2034 | 2236 | 202 |
| 121 | 5 | 2438 | 2640 | 2842 | 3044 | 3246 | 3447 | 3649 | 3850 | 4051 | 4253 | 202 |
| 141 | 6 | 4454 | 4655 | 4856 | 5057 | 5257 | 5458 | 5658 | 5859 | 6059 | 6260 | 201 |
| 162 | 7 | 6460 | 6660 | 6860 | 7060 | 7260 | 7459 | 7659 | 7858 | 8058 | 8257 | 200 |
| 182 | 8 | 8466 | 8666 | 8865 | 9064 | 9263 | 9461 | 9659 | 9859 | 340047 | 340246 | 199 |
| 202 | 9 | 340444 | 340642 | 340841 | 341039 | 341237 | 341435 | 341632 | 341830 | 2028 | 2225 | 198 |

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|-----|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|-----|
| | 220 | 342423 | 342620 | 342817 | 343014 | 343212 | 343409 | 343606 | 343802 | 343999 | 344196 | 197 |
| 19 | 1 | 4392 | 4589 | 4785 | 4981 | 5178 | 5374 | 5570 | 5766 | 5962 | 6157 | 196 |
| 39 | 2 | 6353 | 6549 | 6744 | 6939 | 7135 | 7330 | 7525 | 7720 | 7915 | 8110 | 195 |
| 58 | 3 | 8305 | 8500 | 8694 | 8889 | 9083 | 9278 | 9472 | 9666 | 9860 | 350054 | 194 |
| 77 | 4 | 350248 | 350442 | 350636 | 350829 | 351023 | 351216 | 351410 | 351603 | 351796 | 199 | 193 |
| 97 | 5 | 2183 | 2375 | 2568 | 2761 | 2954 | 3147 | 3339 | 3532 | 3724 | 3916 | 193 |
| 116 | 6 | 4108 | 4291 | 4483 | 4675 | 4876 | 5068 | 5260 | 5452 | 5643 | 5834 | 192 |
| 135 | 7 | 6025 | 6217 | 6408 | 6599 | 6790 | 6981 | 7172 | 7363 | 7554 | 7744 | 191 |
| 154 | 8 | 7935 | 8125 | 8316 | 8506 | 8696 | 8886 | 9076 | 9266 | 9456 | 9646 | 190 |
| 174 | 9 | 9837 | 39025 | 390215 | 390404 | 390593 | 390783 | 390972 | 361161 | 361350 | 361539 | 189 |
| | 230 | 361728 | 361917 | 362105 | 362294 | 362482 | 362671 | 362859 | 363048 | 363236 | 363424 | 189 |
| 19 | 1 | 3612 | 3800 | 3988 | 4176 | 4363 | 4551 | 4739 | 4926 | 5113 | 5301 | 188 |
| 37 | 2 | 5488 | 5675 | 5862 | 6049 | 6236 | 6423 | 6610 | 6796 | 6983 | 7169 | 187 |
| 56 | 3 | 7356 | 7542 | 7729 | 7915 | 8101 | 8287 | 8473 | 8659 | 8845 | 9030 | 186 |
| 74 | 4 | 9216 | 9401 | 9587 | 9772 | 9958 | 370143 | 370328 | 370513 | 370698 | 370883 | 185 |
| 93 | 5 | 370688 | 371273 | 371857 | 372442 | 373026 | 1901 | 2175 | 2360 | 2544 | 2728 | 184 |
| 111 | 6 | 4884 | 5069 | 5253 | 5438 | 5622 | 5806 | 6015 | 6198 | 6382 | 6565 | 184 |
| 130 | 7 | 4718 | 4932 | 5115 | 5298 | 5481 | 5664 | 5846 | 6029 | 6212 | 6394 | 183 |
| 148 | 8 | 6577 | 6759 | 6942 | 7124 | 7306 | 7488 | 7670 | 7852 | 8034 | 8216 | 182 |
| 167 | 9 | 8388 | 8590 | 8761 | 8943 | 9124 | 9306 | 9487 | 9668 | 9849 | 380030 | 181 |
| | 240 | 380211 | 380392 | 380573 | 380754 | 380934 | 381115 | 381296 | 381476 | 381656 | 381837 | 181 |
| 18 | 1 | 2017 | 2197 | 2377 | 2557 | 2737 | 2917 | 3097 | 3277 | 3456 | 3636 | 180 |
| 35 | 2 | 3815 | 3995 | 4174 | 4353 | 4533 | 4712 | 4891 | 5070 | 5249 | 5428 | 179 |
| 53 | 3 | 5696 | 5875 | 5964 | 6142 | 6321 | 6499 | 6677 | 6856 | 7034 | 7212 | 178 |
| 71 | 4 | 7591 | 7769 | 7946 | 8123 | 8301 | 8479 | 8656 | 8834 | 9011 | 9189 | 177 |
| 89 | 5 | 9493 | 9670 | 9847 | 10023 | 10200 | 10377 | 10553 | 10730 | 10906 | 11083 | 176 |
| 106 | 6 | 390935 | 391112 | 391288 | 391464 | 391641 | 391817 | 1993 | 390495 | 390672 | 390849 | 177 |
| 124 | 7 | 2697 | 2873 | 3048 | 3224 | 3400 | 3575 | 3751 | 3926 | 4101 | 4277 | 176 |
| 142 | 8 | 4452 | 4627 | 4802 | 4977 | 5152 | 5326 | 5501 | 5676 | 5850 | 6025 | 175 |
| 159 | 9 | 6199 | 6374 | 6548 | 6722 | 6896 | 7071 | 7245 | 7419 | 7592 | 7766 | 174 |
| | 250 | 397940 | 398114 | 398287 | 398461 | 398634 | 398808 | 398981 | 399154 | 399328 | 399501 | 173 |
| 17 | 1 | 9674 | 9847 | 400120 | 400192 | 400365 | 400538 | 400711 | 400883 | 401056 | 401228 | 173 |
| 34 | 2 | 101401 | 401573 | 1745 | 1917 | 2089 | 2261 | 2433 | 2605 | 2777 | 2949 | 172 |
| 51 | 3 | 3121 | 3292 | 3464 | 3635 | 3807 | 3978 | 4149 | 4320 | 4492 | 4663 | 171 |
| 68 | 4 | 4884 | 5065 | 5246 | 5426 | 5607 | 5787 | 5968 | 6148 | 6328 | 6508 | 171 |
| 85 | 5 | 6540 | 6719 | 6898 | 7077 | 7256 | 7435 | 7614 | 7793 | 7971 | 8150 | 170 |
| 102 | 6 | 8240 | 8419 | 8597 | 8776 | 8954 | 9133 | 9311 | 9489 | 9667 | 9845 | 169 |
| 119 | 7 | 9933 | 410102 | 410271 | 410440 | 410609 | 410777 | 410946 | 411114 | 411283 | 411451 | 169 |
| 136 | 8 | 411620 | 1788 | 1956 | 2124 | 2293 | 2461 | 2629 | 2796 | 2964 | 3132 | 168 |
| 153 | 9 | 3309 | 3467 | 3635 | 3803 | 3970 | 4137 | 4305 | 4472 | 4639 | 4806 | 167 |
| | 260 | 414973 | 415140 | 415307 | 415474 | 415641 | 415808 | 415974 | 416141 | 416308 | 416474 | 167 |
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| 79 | 5 | 9333 | 9491 | 9649 | 9806 | 9964 | 440122 | 440279 | 440437 | 440594 | 440752 | 158 |
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| 41 | 3 | 5544 | 5683 | 5822 | 5960 | 6099 | 6238 | 6376 | 6515 | 6653 | 6791 | 139 |
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| 12 | 1 | 9371 | 9491 | 9608 | 9725 | 9842 | 9959 | 570076 | 570193 | 570309 | 570426 | 117 |
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| 80 | 8 | 660865 | 0960 | 1055 | 1150 | 1245 | 1339 | 1434 | 1529 | 1623 | 1718 | 95 |
| 90 | 9 | 1813 | 1907 | 2002 | 2096 | 2191 | 2286 | 2380 | 2475 | 2569 | 2663 | 95 |

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| 9 | 1 | 3701 | 3795 | 3889 | 3983 | 4078 | 4172 | 4266 | 4360 | 4454 | 4548 | 94 |
| 19 | 2 | 4642 | 4736 | 4830 | 4924 | 5018 | 5112 | 5206 | 5299 | 5393 | 5487 | 94 |
| 28 | 3 | 5581 | 5675 | 5769 | 5862 | 5956 | 6050 | 6143 | 6237 | 6331 | 6424 | 94 |
| 38 | 4 | 6518 | 6612 | 6705 | 6799 | 6892 | 6986 | 7079 | 7173 | 7266 | 7360 | 94 |
| 47 | 5 | 7453 | 7546 | 7640 | 7733 | 7826 | 7920 | 8013 | 8106 | 8199 | 8293 | 93 |
| 56 | 6 | 8386 | 8479 | 8572 | 8665 | 8759 | 8852 | 8945 | 9038 | 9131 | 9224 | 93 |
| 66 | 7 | 9317 | 9410 | 9503 | 9596 | 9689 | 9782 | 9875 | 9967 | 670060 | 670153 | 93 |
| 75 | 8 | 670246 | 670339 | 670431 | 670524 | 670617 | 670710 | 670802 | 670895 | 0988 | 1080 | 93 |
| 85 | 9 | 1173 | 1265 | 1358 | 1451 | 1543 | 1636 | 1729 | 1821 | 1913 | 2005 | 93 |
| | 470 | 672098 | 672190 | 672283 | 672375 | 672467 | 672560 | 672652 | 672744 | 672836 | 672929 | 92 |
| 9 | 1 | 3021 | 3113 | 3205 | 3297 | 3390 | 3482 | 3574 | 3666 | 3758 | 3850 | 92 |
| 18 | 2 | 3942 | 4034 | 4126 | 4218 | 4310 | 4402 | 4494 | 4586 | 4677 | 4769 | 92 |
| 28 | 3 | 4861 | 4953 | 5045 | 5137 | 5228 | 5320 | 5412 | 5503 | 5595 | 5687 | 92 |
| 37 | 4 | 5778 | 5870 | 5962 | 6053 | 6145 | 6236 | 6328 | 6419 | 6511 | 6602 | 92 |
| 46 | 5 | 6694 | 6785 | 6876 | 6968 | 7059 | 7151 | 7242 | 7333 | 7424 | 7516 | 91 |
| 55 | 6 | 7607 | 7698 | 7789 | 7881 | 7972 | 8063 | 8154 | 8245 | 8336 | 8427 | 91 |
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| | 480 | 681211 | 681332 | 681422 | 681513 | 681603 | 681693 | 681784 | 681874 | 681964 | 682055 | 90 |
| 9 | 1 | 2145 | 2235 | 2326 | 2416 | 2506 | 2596 | 2686 | 2777 | 2867 | 2957 | 90 |
| 18 | 2 | 3047 | 3137 | 3227 | 3317 | 3407 | 3497 | 3587 | 3677 | 3767 | 3857 | 90 |
| 27 | 3 | 3947 | 4037 | 4127 | 4217 | 4307 | 4396 | 4486 | 4576 | 4666 | 4756 | 90 |
| 36 | 4 | 4845 | 4935 | 5025 | 5114 | 5204 | 5294 | 5383 | 5473 | 5563 | 5652 | 90 |
| 45 | 5 | 5742 | 5831 | 5921 | 6010 | 6100 | 6189 | 6279 | 6368 | 6458 | 6547 | 89 |
| 54 | 6 | 6639 | 6728 | 6818 | 6907 | 6997 | 7086 | 7175 | 7265 | 7354 | 7443 | 89 |
| 63 | 7 | 7529 | 7618 | 7707 | 7796 | 7886 | 7975 | 8064 | 8153 | 8242 | 8331 | 89 |
| 72 | 8 | 8420 | 8509 | 8598 | 8687 | 8776 | 8865 | 8953 | 9042 | 9131 | 9220 | 89 |
| 81 | 9 | 9309 | 9398 | 9486 | 9575 | 9664 | 9753 | 9841 | 9930 | 690019 | 690107 | 89 |
| | 490 | 690196 | 690285 | 690373 | 690462 | 690550 | 690639 | 690728 | 690816 | 690905 | 690993 | 89 |
| 9 | 1 | 1081 | 1170 | 1258 | 1347 | 1435 | 1524 | 1612 | 1700 | 1789 | 1877 | 88 |
| 18 | 2 | 1965 | 2053 | 2142 | 2230 | 2318 | 2406 | 2494 | 2583 | 2671 | 2759 | 88 |
| 26 | 3 | 2847 | 2935 | 3023 | 3111 | 3199 | 3287 | 3375 | 3463 | 3551 | 3639 | 88 |
| 35 | 4 | 3727 | 3815 | 3903 | 3991 | 4078 | 4166 | 4254 | 4342 | 4430 | 4517 | 88 |
| 44 | 5 | 4905 | 4993 | 4781 | 4868 | 4956 | 5044 | 5131 | 5219 | 5307 | 5394 | 88 |
| 53 | 6 | 5182 | 5269 | 5357 | 5444 | 5532 | 5619 | 5707 | 5794 | 5882 | 5969 | 87 |
| 62 | 7 | 6356 | 6444 | 6531 | 6618 | 6706 | 6793 | 6880 | 6968 | 7055 | 7142 | 87 |
| 70 | 8 | 7224 | 7311 | 7404 | 7491 | 7578 | 7665 | 7752 | 7839 | 7926 | 8014 | 87 |
| 79 | 9 | 8101 | 8188 | 8275 | 8362 | 8449 | 8535 | 8622 | 8709 | 8796 | 8883 | 87 |
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| 9 | 1 | 9838 | 9924 | 700011 | 700098 | 700184 | 700271 | 700358 | 700444 | 700531 | 700617 | 87 |
| 17 | 2 | 700704 | 700790 | 0877 | 0963 | 1050 | 1136 | 1222 | 1309 | 1395 | 1482 | 86 |
| 26 | 3 | 1568 | 1654 | 1741 | 1827 | 1913 | 1999 | 2086 | 2172 | 2258 | 2344 | 86 |
| 34 | 4 | 2431 | 2517 | 2603 | 2689 | 2775 | 2861 | 2947 | 3033 | 3119 | 3205 | 86 |
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| 52 | 6 | 4154 | 4239 | 4322 | 4408 | 4494 | 4579 | 4665 | 4751 | 4837 | 4922 | 86 |
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| 69 | 8 | 5864 | 5949 | 6035 | 6120 | 6206 | 6291 | 6376 | 6462 | 6547 | 6632 | 85 |
| 77 | 9 | 6718 | 6803 | 6888 | 6974 | 7059 | 7144 | 7229 | 7315 | 7400 | 7485 | 85 |
| | 510 | 707570 | 707655 | 707740 | 707826 | 707911 | 707996 | 708081 | 708166 | 708251 | 708336 | 85 |
| 8 | 1 | 8421 | 8506 | 8591 | 8676 | 8761 | 8846 | 8931 | 9015 | 9100 | 9185 | 85 |
| 17 | 2 | 9279 | 9355 | 9440 | 9524 | 9609 | 9694 | 9779 | 9863 | 9948 | 710033 | 85 |
| 25 | 3 | 710117 | 710202 | 710287 | 710371 | 710456 | 710540 | 710625 | 710710 | 710794 | 0879 | 85 |
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| 67 | 8 | 4330 | 4414 | 4497 | 4581 | 4665 | 4749 | 4833 | 4916 | 5000 | 5084 | 84 |
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| 8 | 1 | 6838 | 6921 | 7004 | 7088 | 7171 | 7254 | 7338 | 7421 | 7504 | 7587 | 83 |
| 17 | 2 | 7671 | 7754 | 7837 | 7920 | 8003 | 8086 | 8169 | 8253 | 8336 | 8419 | 83 |
| 25 | 3 | 8602 | 8685 | 8768 | 8851 | 8934 | 9017 | 9100 | 9183 | 9266 | 9349 | 83 |
| 33 | 4 | 9331 | 9414 | 9497 | 9580 | 9663 | 9746 | 9829 | 9912 | 9995 | 720677 | 83 |
| 41 | 5 | 720159 | 720212 | 720265 | 720317 | 720370 | 720423 | 720475 | 720528 | 720581 | 720634 | 83 |
| 50 | 6 | 0986 | 1068 | 1151 | 1233 | 1316 | 1398 | 1481 | 1563 | 1646 | 1728 | 82 |
| 58 | 7 | 1811 | 1893 | 1975 | 2058 | 2140 | 2222 | 2305 | 2387 | 2469 | 2552 | 82 |
| 66 | 8 | 2634 | 2716 | 2798 | 2881 | 2963 | 3045 | 3127 | 3209 | 3291 | 3374 | 82 |
| 75 | 9 | 3456 | 3538 | 3620 | 3702 | 3784 | 3866 | 3948 | 4030 | 4112 | 4194 | 82 |
| 530 | 724276 | 724358 | 724440 | 724522 | 724604 | 724685 | 724767 | 724849 | 724931 | 725013 | 82 | |
| 8 | 1 | 5093 | 5176 | 5258 | 5340 | 5422 | 5503 | 5585 | 5667 | 5748 | 5830 | 82 |
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| 57 | 7 | 9974 | 73005 | 730136 | 730217 | 730298 | 730379 | 730459 | 730540 | 730621 | 730702 | 81 |
| 65 | 8 | 730782 | 0863 | 0944 | 1024 | 1105 | 1186 | 1266 | 1347 | 1428 | 1508 | 81 |
| 73 | 9 | 1589 | 1669 | 1750 | 1830 | 1911 | 1991 | 2072 | 2152 | 2233 | 2313 | 81 |
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| 8 | 1 | 3197 | 3278 | 3358 | 3438 | 3518 | 3598 | 3679 | 3759 | 3839 | 3919 | 80 |
| 16 | 2 | 3999 | 4079 | 4160 | 4240 | 4320 | 4400 | 4480 | 4560 | 4640 | 4720 | 80 |
| 24 | 3 | 4800 | 4880 | 4960 | 5040 | 5120 | 5200 | 5279 | 5359 | 5439 | 5519 | 80 |
| 32 | 4 | 5529 | 5609 | 5689 | 5769 | 5848 | 5928 | 6008 | 6087 | 6167 | 6247 | 80 |
| 41 | 5 | 6397 | 6476 | 6556 | 6635 | 6715 | 6795 | 6874 | 6954 | 7034 | 7113 | 80 |
| 49 | 6 | 7193 | 7272 | 7352 | 7431 | 7511 | 7590 | 7670 | 7749 | 7829 | 7908 | 79 |
| 58 | 7 | 7987 | 8067 | 8146 | 8225 | 8305 | 8384 | 8463 | 8543 | 8622 | 8701 | 79 |
| 64 | 8 | 8781 | 8860 | 8939 | 9018 | 9097 | 9177 | 9256 | 9335 | 9414 | 9493 | 79 |
| 72 | 9 | 9572 | 9651 | 9731 | 9810 | 9889 | 9968 | 740047 | 740126 | 740205 | 740284 | 79 |
| 550 | 740363 | 740442 | 740521 | 740600 | 740678 | 740757 | 740836 | 740915 | 740994 | 741073 | 79 | |
| 8 | 1 | 1152 | 1230 | 1309 | 1388 | 1467 | 1546 | 1624 | 1703 | 1782 | 1860 | 79 |
| 16 | 2 | 1939 | 2018 | 2096 | 2175 | 2254 | 2332 | 2411 | 2489 | 2568 | 2647 | 79 |
| 23 | 3 | 2735 | 2804 | 2882 | 2961 | 3039 | 3118 | 3196 | 3275 | 3354 | 3433 | 78 |
| 31 | 4 | 3510 | 3588 | 3667 | 3745 | 3823 | 3902 | 3980 | 4058 | 4136 | 4215 | 78 |
| 39 | 5 | 4293 | 4371 | 4449 | 4528 | 4606 | 4684 | 4762 | 4840 | 4919 | 4997 | 78 |
| 47 | 6 | 5075 | 5153 | 5231 | 5309 | 5387 | 5465 | 5543 | 5621 | 5699 | 5777 | 78 |
| 55 | 7 | 5855 | 5933 | 6011 | 6089 | 6167 | 6245 | 6323 | 6401 | 6479 | 6556 | 78 |
| 62 | 8 | 6634 | 6712 | 6790 | 6868 | 6945 | 7023 | 7101 | 7179 | 7256 | 7334 | 78 |
| 70 | 9 | 7412 | 7489 | 7567 | 7645 | 7722 | 7800 | 7878 | 7955 | 8033 | 8110 | 78 |
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| 8 | 1 | 8063 | 8040 | 8118 | 8195 | 8272 | 8350 | 8427 | 8504 | 8582 | 8659 | 77 |
| 15 | 2 | 9736 | 9814 | 9891 | 9968 | 750045 | 750122 | 750200 | 750277 | 750354 | 750431 | 77 |
| 23 | 3 | 750508 | 750586 | 750663 | 750740 | 0817 | 0894 | 0971 | 1048 | 1125 | 1202 | 77 |
| 31 | 4 | 1279 | 1356 | 1433 | 1510 | 1587 | 1664 | 1741 | 1818 | 1895 | 1972 | 77 |
| 39 | 5 | 2048 | 2125 | 2202 | 2279 | 2356 | 2433 | 2509 | 2586 | 2663 | 2740 | 77 |
| 46 | 6 | 2816 | 2893 | 2970 | 3047 | 3123 | 3200 | 3277 | 3353 | 3430 | 3506 | 77 |
| 54 | 7 | 3583 | 3660 | 3736 | 3813 | 3889 | 3966 | 4042 | 4119 | 4195 | 4272 | 77 |
| 62 | 8 | 4348 | 4425 | 4501 | 4578 | 4654 | 4731 | 4807 | 4883 | 4960 | 5036 | 76 |
| 69 | 9 | 5112 | 6189 | 6265 | 6341 | 5117 | 5194 | 5270 | 5646 | 5722 | 5799 | 76 |
| 570 | 755975 | 755951 | 755927 | 755903 | 755879 | 755855 | 755832 | 755808 | 755784 | 755760 | 76 | |
| 8 | 1 | 6391 | 6712 | 6788 | 6864 | 6940 | 7016 | 7092 | 7168 | 7244 | 7320 | 76 |
| 15 | 2 | 7386 | 7472 | 7548 | 7624 | 7700 | 7775 | 7851 | 7927 | 8003 | 8079 | 76 |
| 23 | 3 | 8155 | 8250 | 8306 | 8362 | 8418 | 8473 | 8528 | 8583 | 8638 | 8693 | 76 |
| 30 | 4 | 8912 | 8988 | 9063 | 9139 | 9214 | 9290 | 9366 | 9441 | 9517 | 9592 | 76 |
| 38 | 5 | 9688 | 9743 | 9819 | 9894 | 9970 | 760045 | 760121 | 760196 | 760272 | 760347 | 76 |
| 46 | 6 | 760422 | 760498 | 760573 | 760649 | 760724 | 0799 | 0875 | 0950 | 1025 | 1101 | 75 |
| 53 | 7 | 1176 | 1251 | 1326 | 1402 | 1477 | 1552 | 1627 | 1702 | 1777 | 1853 | 75 |
| 61 | 8 | 1928 | 2003 | 2078 | 2153 | 2228 | 2303 | 2378 | 2453 | 2529 | 2604 | 75 |
| 68 | 9 | 2679 | 2754 | 2829 | 2904 | 2978 | 3053 | 3128 | 3203 | 3278 | 3353 | 75 |

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| 7 | 1 | 4176 | 4251 | 4326 | 4400 | 4475 | 4550 | 4624 | 4699 | 4774 | 4848 | 75 |
| 15 | 2 | 4922 | 4997 | 5072 | 5147 | 5221 | 5296 | 5370 | 5445 | 5520 | 5594 | 75 |
| 22 | 3 | 5669 | 5744 | 5818 | 5892 | 5966 | 6041 | 6115 | 6190 | 6264 | 6338 | 74 |
| 29 | 4 | 6415 | 6490 | 6564 | 6638 | 6712 | 6785 | 6859 | 6933 | 7007 | 7082 | 74 |
| 37 | 5 | 7161 | 7235 | 7309 | 7383 | 7457 | 7530 | 7604 | 7678 | 7752 | 7826 | 74 |
| 44 | 6 | 7908 | 7982 | 8056 | 8129 | 8203 | 8277 | 8350 | 8424 | 8498 | 8572 | 74 |
| 52 | 7 | 8654 | 8728 | 8802 | 8876 | 8949 | 9023 | 9097 | 9170 | 9244 | 9318 | 74 |
| 59 | 8 | 9377 | 9451 | 9525 | 9599 | 9673 | 9746 | 9820 | 9894 | 9968 | 70042 | 74 |
| 67 | 9 | 770115 | 770189 | 770263 | 770337 | 770410 | 770484 | 770557 | 770631 | 770705 | 0778 | 74 |
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| 7 | 1 | 1587 | 1661 | 1734 | 1808 | 1881 | 1955 | 2028 | 2102 | 2175 | 2248 | 73 |
| 15 | 2 | 2332 | 2406 | 2479 | 2552 | 2625 | 2698 | 2771 | 2845 | 2918 | 2991 | 73 |
| 22 | 3 | 3077 | 3150 | 3223 | 3296 | 3369 | 3442 | 3515 | 3588 | 3661 | 3734 | 73 |
| 29 | 4 | 3783 | 3856 | 3929 | 4002 | 4075 | 4148 | 4221 | 4294 | 4367 | 4440 | 73 |
| 37 | 5 | 4517 | 4590 | 4663 | 4736 | 4809 | 4882 | 4955 | 5028 | 5101 | 5174 | 73 |
| 44 | 6 | 5261 | 5334 | 5407 | 5480 | 5553 | 5626 | 5699 | 5772 | 5845 | 5918 | 73 |
| 52 | 7 | 5974 | 6047 | 6120 | 6193 | 6266 | 6339 | 6412 | 6485 | 6558 | 6631 | 73 |
| 58 | 8 | 6701 | 6774 | 6847 | 6920 | 6993 | 7066 | 7139 | 7212 | 7285 | 7358 | 73 |
| 66 | 9 | 7427 | 7500 | 7573 | 7646 | 7719 | 7792 | 7865 | 7938 | 8011 | 8084 | 72 |
| | 600 | 778151 | 778224 | 778297 | 778370 | 778443 | 778516 | 778589 | 778662 | 778735 | 778808 | 72 |
| 7 | 1 | 8874 | 8947 | 9020 | 9093 | 9166 | 9239 | 9312 | 9385 | 9458 | 9531 | 72 |
| 14 | 2 | 9596 | 9669 | 9742 | 9815 | 9888 | 9961 | 780029 | 780101 | 780173 | 780245 | 72 |
| 22 | 3 | 780317 | 780389 | 780461 | 780533 | 780605 | 780677 | 0749 | 0821 | 0893 | 0965 | 72 |
| 29 | 4 | 1037 | 1109 | 1181 | 1253 | 1324 | 1396 | 1468 | 1540 | 1612 | 1684 | 72 |
| 36 | 5 | 1753 | 1825 | 1897 | 1969 | 2041 | 2113 | 2185 | 2257 | 2329 | 2401 | 72 |
| 43 | 6 | 2470 | 2542 | 2614 | 2686 | 2758 | 2830 | 2902 | 2974 | 3046 | 3117 | 72 |
| 50 | 7 | 3189 | 3261 | 3333 | 3405 | 3477 | 3549 | 3621 | 3693 | 3765 | 3837 | 71 |
| 58 | 8 | 3904 | 3976 | 4048 | 4120 | 4192 | 4264 | 4336 | 4408 | 4480 | 4552 | 71 |
| 65 | 9 | 4617 | 4689 | 4761 | 4833 | 4905 | 4977 | 5049 | 5121 | 5193 | 5265 | 71 |
| | 610 | 785330 | 785401 | 785472 | 785543 | 785615 | 785686 | 785757 | 785828 | 785899 | 785970 | 71 |
| 7 | 1 | 6041 | 6112 | 6183 | 6254 | 6325 | 6396 | 6467 | 6538 | 6609 | 6680 | 71 |
| 14 | 2 | 6751 | 6822 | 6893 | 6964 | 7035 | 7106 | 7177 | 7248 | 7319 | 7390 | 71 |
| 21 | 3 | 7460 | 7531 | 7602 | 7673 | 7744 | 7815 | 7886 | 7957 | 8027 | 8098 | 71 |
| 28 | 4 | 8168 | 8239 | 8310 | 8381 | 8452 | 8523 | 8594 | 8665 | 8736 | 8807 | 71 |
| 35 | 5 | 8875 | 8946 | 9017 | 9088 | 9159 | 9229 | 9300 | 9371 | 9442 | 9513 | 71 |
| 43 | 6 | 9584 | 9655 | 9726 | 9797 | 9868 | 9939 | 790004 | 790074 | 790144 | 790215 | 70 |
| 50 | 7 | 790285 | 790356 | 790426 | 790496 | 790567 | 790637 | 0707 | 0778 | 0848 | 0918 | 70 |
| 57 | 8 | 0988 | 1059 | 1129 | 1199 | 1269 | 1340 | 1410 | 1480 | 1550 | 1620 | 70 |
| 64 | 9 | 1691 | 1761 | 1831 | 1901 | 1971 | 2041 | 2111 | 2181 | 2252 | 2322 | 70 |
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| 35 | 5 | 5889 | 5958 | 6028 | 6098 | 6168 | 6237 | 6307 | 6376 | 6446 | 6515 | 69 |
| 42 | 6 | 6574 | 6644 | 6713 | 6782 | 6852 | 6921 | 6990 | 7060 | 7129 | 7198 | 69 |
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| 56 | 8 | 7960 | 8029 | 8098 | 8167 | 8236 | 8305 | 8374 | 8443 | 8513 | 8582 | 69 |
| 63 | 9 | 8651 | 8720 | 8789 | 8858 | 8927 | 8996 | 9065 | 9134 | 9203 | 9272 | 69 |
| | 630 | 799311 | 799409 | 799478 | 799547 | 799616 | 799685 | 799754 | 799823 | 799892 | 799961 | 69 |
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| 14 | 2 | 0717 | 0786 | 0854 | 0923 | 0992 | 1061 | 1129 | 1198 | 1266 | 1335 | 69 |
| 21 | 3 | 1404 | 1472 | 1541 | 1609 | 1678 | 1747 | 1815 | 1884 | 1952 | 2021 | 69 |
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| 41 | 6 | 3457 | 3525 | 3594 | 3662 | 3730 | 3798 | 3867 | 3935 | 4003 | 4071 | 68 |
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| 47 | 7 | 0904 | 0971 | 1039 | 1106 | 1173 | 1240 | 1307 | 1374 | 1441 | 1508 | 67 |
| 54 | 8 | 1575 | 1642 | 1709 | 1776 | 1843 | 1910 | 1977 | 2044 | 2111 | 2178 | 67 |
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| 53 | 8 | 8226 | 8292 | 8358 | 8424 | 8490 | 8556 | 8622 | 8688 | 8754 | 8820 | 66 |
| 59 | 9 | 8885 | 8951 | 9017 | 9083 | 9149 | 9215 | 9281 | 9346 | 9412 | 9478 | 66 |
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| 13 | 2 | 0858 | 0924 | 0989 | 1055 | 1120 | 1186 | 1251 | 1317 | 1382 | 1448 | 66 |
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| 6 | 1 | 6723 | 6787 | 6852 | 6917 | 6981 | 7046 | 7111 | 7175 | 7240 | 7305 | 65 |
| 13 | 2 | 7369 | 7434 | 7499 | 7563 | 7628 | 7692 | 7757 | 7821 | 7886 | 7951 | 65 |
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| 58 | 9 | 1870 | 1934 | 1998 | 2062 | 2126 | 2189 | 2253 | 2317 | 2381 | 2445 | 64 |
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| 13 | 2 | 3781 | 3845 | 3908 | 3972 | 4035 | 4100 | 4164 | 4228 | 4291 | 4355 | 64 |
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| 19 | 3 | 0733 | 0796 | 0859 | 0922 | 0984 | 1047 | 1110 | 1172 | 1235 | 1297 | 63 |
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| 54 | 9 | 2725 | 2787 | 2847 | 2906 | 2966 | 3025 | 3085 | 3144 | 3204 | 3263 | 60 |
| | 730 | 863323 | 863382 | 863442 | 863501 | 863561 | 863620 | 863679 | 863739 | 863799 | 863858 | 59 |
| 6 | 1 | 3917 | 3977 | 4036 | 4096 | 4155 | 4214 | 4274 | 4333 | 4392 | 4452 | 59 |
| 12 | 2 | 4511 | 4570 | 4630 | 4689 | 4748 | 4808 | 4867 | 4926 | 4985 | 5045 | 59 |
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| 30 | 5 | 6287 | 6346 | 6405 | 6465 | 6524 | 6583 | 6642 | 6701 | 6760 | 6819 | 59 |
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| 41 | 7 | 7467 | 7526 | 7585 | 7644 | 7703 | 7762 | 7821 | 7880 | 7939 | 7998 | 59 |
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| 11 | 2 | 1955 | 2012 | 2069 | 2126 | 2183 | 2240 | 2297 | 2354 | 2411 | 2468 | 57 |
| 17 | 3 | 2625 | 2682 | 2739 | 2796 | 2853 | 2910 | 2967 | 3024 | 3081 | 3138 | 57 |
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| 23 | 4 | 8741 | 8798 | 8855 | 8912 | 8969 | 9026 | 9083 | 9140 | 9197 | 9254 | 56 |
| 28 | 5 | 9302 | 9359 | 9416 | 9473 | 9530 | 9587 | 9644 | 9701 | 9758 | 9815 | 56 |
| 34 | 6 | 9862 | 9919 | 9976 | 10033 | 10090 | 10147 | 10204 | 10261 | 10318 | 10375 | 56 |
| 39 | 7 | 890421 | 890477 | 890533 | 890589 | 890645 | 890701 | 890757 | 890813 | 890869 | 890925 | 56 |
| 45 | 8 | 0980 | 1035 | 1090 | 1145 | 1200 | 1255 | 1310 | 1365 | 1420 | 1475 | 56 |
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| 790 | 1 | 89707 | 89762 | 89817 | 89872 | 89927 | 89982 | 90037 | 90092 | 90147 | 90202 | 56 |
| 5 | 1 | 8176 | 8231 | 8286 | 8341 | 8396 | 8451 | 8506 | 8561 | 8616 | 8671 | 56 |
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| 22 | 4 | 5256 | 5310 | 5364 | 5418 | 5472 | 5526 | 5580 | 5634 | 5688 | 5742 | 54 |
| 27 | 5 | 5796 | 5850 | 5904 | 5958 | 6012 | 6066 | 6119 | 6173 | 6227 | 6281 | 54 |
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| 36 | 7 | 7883 | 7935 | 7986 | 8037 | 8088 | 8140 | 8191 | 8242 | 8293 | 8345 | 51 |
| 41 | 8 | 8396 | 8447 | 8498 | 8549 | 8601 | 8652 | 8703 | 8754 | 8805 | 8857 | 51 |
| 46 | 9 | 8908 | 8959 | 9010 | 9061 | 9112 | 9163 | 9215 | 9266 | 9317 | 9368 | 51 |
| | 850 | 929419 | 929470 | 929521 | 929572 | 929623 | 929674 | 929725 | 929776 | 929827 | 929879 | 51 |
| 5 | 1 | 9930 | 9981 | 9993 | 9993 | 9993 | 9993 | 9993 | 9993 | 9993 | 9993 | 51 |
| 10 | 2 | 93040 | 93040 | 93040 | 93040 | 93040 | 93040 | 93040 | 93040 | 93040 | 93040 | 51 |
| 15 | 3 | 0949 | 1000 | 1051 | 1102 | 1153 | 1204 | 1254 | 1305 | 1356 | 1407 | 51 |
| 20 | 4 | 1478 | 1529 | 1580 | 1630 | 1681 | 1731 | 1782 | 1833 | 1884 | 1935 | 51 |
| 25 | 5 | 1966 | 2017 | 2068 | 2118 | 2169 | 2220 | 2271 | 2322 | 2372 | 2423 | 51 |
| 31 | 6 | 2474 | 2524 | 2575 | 2626 | 2677 | 2727 | 2778 | 2829 | 2879 | 2930 | 51 |
| 36 | 7 | 2981 | 3031 | 3082 | 3133 | 3183 | 3234 | 3285 | 3335 | 3386 | 3437 | 51 |
| 41 | 8 | 3487 | 3538 | 3589 | 3639 | 3690 | 3740 | 3791 | 3841 | 3892 | 3943 | 51 |
| 46 | 9 | 3993 | 4044 | 4094 | 4145 | 4195 | 4246 | 4296 | 4347 | 4397 | 4448 | 51 |
| | 860 | 934498 | 934549 | 934599 | 934650 | 934700 | 934751 | 934801 | 934852 | 934902 | 934953 | 50 |
| 5 | 1 | 5093 | 5054 | 5104 | 5154 | 5205 | 5255 | 5306 | 5356 | 5406 | 5457 | 50 |
| 10 | 2 | 5507 | 5558 | 5608 | 5658 | 5709 | 5759 | 5809 | 5859 | 5910 | 5960 | 50 |
| 15 | 3 | 6011 | 6061 | 6111 | 6162 | 6212 | 6262 | 6313 | 6363 | 6413 | 6463 | 50 |
| 20 | 4 | 6514 | 6564 | 6614 | 6665 | 6715 | 6765 | 6815 | 6865 | 6916 | 6966 | 50 |
| 25 | 5 | 7016 | 7066 | 7117 | 7167 | 7217 | 7267 | 7317 | 7367 | 7418 | 7468 | 50 |
| 30 | 6 | 7518 | 7568 | 7618 | 7668 | 7718 | 7769 | 7819 | 7869 | 7919 | 7969 | 50 |
| 35 | 7 | 8019 | 8069 | 8119 | 8169 | 8219 | 8269 | 8320 | 8370 | 8420 | 8470 | 50 |
| 40 | 8 | 8520 | 8570 | 8620 | 8670 | 8720 | 8770 | 8820 | 8870 | 8920 | 8970 | 50 |
| 45 | 9 | 9020 | 9070 | 9120 | 9170 | 9220 | 9270 | 9320 | 9370 | 9419 | 9469 | 50 |
| | 870 | 939519 | 939569 | 939619 | 939669 | 939719 | 939769 | 939819 | 939869 | 939919 | 939968 | 50 |
| 5 | 1 | 94018 | 94068 | 94018 | 94018 | 94018 | 94018 | 94018 | 94018 | 94018 | 94018 | 50 |
| 10 | 2 | 0616 | 0566 | 0616 | 0666 | 0716 | 0765 | 0815 | 0865 | 0915 | 0964 | 50 |
| 15 | 3 | 1014 | 1064 | 1114 | 1163 | 1213 | 1263 | 1313 | 1362 | 1412 | 1462 | 50 |
| 20 | 4 | 1511 | 1561 | 1611 | 1660 | 1710 | 1760 | 1809 | 1859 | 1909 | 1958 | 50 |
| 25 | 5 | 2008 | 2058 | 2107 | 2157 | 2207 | 2256 | 2306 | 2355 | 2405 | 2455 | 50 |
| 30 | 6 | 2504 | 2554 | 2603 | 2653 | 2702 | 2752 | 2801 | 2851 | 2901 | 2950 | 50 |
| 35 | 7 | 3000 | 3049 | 3099 | 3148 | 3198 | 3247 | 3297 | 3346 | 3396 | 3445 | 49 |
| 40 | 8 | 3495 | 3544 | 3593 | 3643 | 3692 | 3742 | 3792 | 3841 | 3890 | 3939 | 49 |
| 45 | 9 | 3989 | 4038 | 4088 | 4137 | 4186 | 4236 | 4285 | 4335 | 4384 | 4433 | 49 |

| PP | N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | D. |
|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|----|
| 940 | 973128 | 973174 | 973220 | 973266 | 973313 | 973359 | 973405 | 973451 | 973497 | 973543 | 46 | |
| 5 | 1 | 3590 | 3636 | 3682 | 3728 | 3774 | 3820 | 3866 | 3913 | 3959 | 4005 | 46 |
| 9 | 2 | 4051 | 4097 | 4143 | 4189 | 4235 | 4281 | 4327 | 4374 | 4420 | 4466 | 46 |
| 14 | 3 | 4512 | 4558 | 4604 | 4650 | 4696 | 4742 | 4788 | 4834 | 4880 | 4926 | 46 |
| 18 | 4 | 4972 | 5018 | 5064 | 5110 | 5156 | 5202 | 5248 | 5294 | 5340 | 5386 | 46 |
| 23 | 5 | 5432 | 5478 | 5524 | 5570 | 5616 | 5662 | 5707 | 5753 | 5799 | 5845 | 46 |
| 28 | 6 | 5891 | 5937 | 5983 | 6029 | 6075 | 6121 | 6167 | 6212 | 6258 | 6304 | 46 |
| 32 | 7 | 6360 | 6396 | 6442 | 6488 | 6533 | 6579 | 6625 | 6671 | 6717 | 6763 | 46 |
| 37 | 8 | 6805 | 6841 | 6886 | 6932 | 6977 | 7023 | 7068 | 7113 | 7159 | 7204 | 46 |
| 41 | 9 | 7206 | 7252 | 7298 | 7343 | 7389 | 7435 | 7480 | 7526 | 7571 | 7617 | 46 |
| 950 | 977724 | 977769 | 977815 | 977861 | 977906 | 977952 | 977998 | 978043 | 978089 | 978135 | 46 | |
| 5 | 1 | 8181 | 8226 | 8272 | 8317 | 8363 | 8409 | 8454 | 8500 | 8546 | 8591 | 46 |
| 9 | 2 | 8637 | 8683 | 8728 | 8774 | 8819 | 8865 | 8911 | 8956 | 9002 | 9047 | 46 |
| 14 | 3 | 9093 | 9138 | 9184 | 9230 | 9275 | 9321 | 9366 | 9412 | 9457 | 9503 | 46 |
| 18 | 4 | 9548 | 9594 | 9639 | 9685 | 9730 | 9776 | 9821 | 9867 | 9912 | 9958 | 46 |
| 23 | 5 | 98003 | 98049 | 98094 | 98140 | 98185 | 98231 | 98276 | 98322 | 98367 | 98413 | 45 |
| 27 | 6 | 0458 | 0503 | 0549 | 0594 | 0640 | 0685 | 0730 | 0776 | 0821 | 0867 | 45 |
| 32 | 7 | 0912 | 0957 | 1003 | 1048 | 1093 | 1139 | 1184 | 1229 | 1274 | 1320 | 45 |
| 36 | 8 | 1365 | 1411 | 1456 | 1501 | 1547 | 1592 | 1637 | 1683 | 1728 | 1773 | 45 |
| 41 | 9 | 1819 | 1864 | 1909 | 1954 | 2000 | 2045 | 2090 | 2135 | 2181 | 2226 | 45 |
| 960 | 982271 | 982316 | 982362 | 982407 | 982452 | 982497 | 982543 | 982588 | 982633 | 982678 | 45 | |
| 5 | 1 | 2723 | 2769 | 2814 | 2859 | 2904 | 2949 | 2994 | 3040 | 3085 | 3130 | 45 |
| 9 | 2 | 3175 | 3220 | 3265 | 3310 | 3356 | 3401 | 3446 | 3491 | 3536 | 3581 | 45 |
| 14 | 3 | 3620 | 3665 | 3710 | 3756 | 3801 | 3847 | 3892 | 3937 | 3982 | 4027 | 45 |
| 18 | 4 | 4072 | 4117 | 4162 | 4207 | 4252 | 4297 | 4342 | 4387 | 4432 | 4477 | 45 |
| 23 | 5 | 4527 | 4572 | 4617 | 4662 | 4707 | 4752 | 4797 | 4842 | 4887 | 4932 | 45 |
| 27 | 6 | 4977 | 5022 | 5067 | 5112 | 5157 | 5202 | 5247 | 5292 | 5337 | 5382 | 45 |
| 32 | 7 | 5427 | 5471 | 5516 | 5561 | 5606 | 5651 | 5696 | 5741 | 5786 | 5830 | 45 |
| 36 | 8 | 5875 | 5920 | 5965 | 6010 | 6055 | 6100 | 6144 | 6189 | 6234 | 6279 | 45 |
| 41 | 9 | 6324 | 6369 | 6413 | 6458 | 6503 | 6548 | 6593 | 6637 | 6682 | 6727 | 45 |
| 970 | 986772 | 986817 | 986861 | 986906 | 986951 | 986996 | 987040 | 987085 | 987130 | 987175 | 45 | |
| 5 | 1 | 7219 | 7264 | 7309 | 7353 | 7398 | 7443 | 7488 | 7532 | 7577 | 7622 | 45 |
| 9 | 2 | 7666 | 7711 | 7756 | 7800 | 7845 | 7890 | 7934 | 7979 | 8024 | 8068 | 45 |
| 14 | 3 | 8113 | 8157 | 8202 | 8247 | 8291 | 8336 | 8381 | 8425 | 8470 | 8514 | 45 |
| 18 | 4 | 8559 | 8604 | 8648 | 8693 | 8737 | 8782 | 8826 | 8871 | 8916 | 8960 | 45 |
| 23 | 5 | 9005 | 9049 | 9094 | 9138 | 9183 | 9227 | 9272 | 9316 | 9361 | 9405 | 45 |
| 27 | 6 | 9450 | 9494 | 9539 | 9583 | 9628 | 9672 | 9717 | 9761 | 9806 | 9850 | 44 |
| 32 | 7 | 9895 | 9939 | 9983 | 00028 | 00072 | 00117 | 00161 | 00206 | 00250 | 00294 | 44 |
| 36 | 8 | 99839 | 99883 | 99928 | 00472 | 0516 | 0561 | 0605 | 0650 | 0694 | 0738 | 44 |
| 41 | 9 | 0783 | 0827 | 0871 | 0916 | 0960 | 1004 | 1049 | 1093 | 1137 | 1182 | 44 |
| 980 | 991225 | 991270 | 991315 | 991359 | 991403 | 991448 | 991492 | 991536 | 991580 | 991625 | 991669 | 44 |
| 4 | 1 | 1669 | 1713 | 1758 | 1802 | 1846 | 1890 | 1935 | 1979 | 2023 | 2067 | 44 |
| 9 | 2 | 2111 | 2156 | 2200 | 2244 | 2288 | 2333 | 2377 | 2421 | 2465 | 2509 | 44 |
| 13 | 3 | 2554 | 2598 | 2642 | 2686 | 2730 | 2774 | 2819 | 2863 | 2907 | 2951 | 44 |
| 18 | 4 | 2995 | 3039 | 3083 | 3127 | 3172 | 3216 | 3260 | 3304 | 3348 | 3392 | 44 |
| 22 | 5 | 3436 | 3480 | 3524 | 3568 | 3613 | 3657 | 3701 | 3745 | 3789 | 3833 | 44 |
| 26 | 6 | 3877 | 3921 | 3965 | 4009 | 4053 | 4097 | 4141 | 4185 | 4229 | 4273 | 44 |
| 30 | 7 | 4317 | 4361 | 4405 | 4449 | 4493 | 4537 | 4581 | 4625 | 4669 | 4713 | 44 |
| 35 | 8 | 4757 | 4801 | 4845 | 4889 | 4933 | 4977 | 5021 | 5065 | 5108 | 5152 | 44 |
| 40 | 9 | 5196 | 5240 | 5284 | 5328 | 5372 | 5416 | 5460 | 5504 | 5547 | 5591 | 44 |
| 990 | 995635 | 995679 | 995723 | 995767 | 995811 | 995854 | 995898 | 995942 | 995986 | 996030 | 44 | |
| 4 | 1 | 6074 | 6117 | 6161 | 6205 | 6249 | 6293 | 6337 | 6380 | 6424 | 6468 | 44 |
| 9 | 2 | 6512 | 6555 | 6599 | 6643 | 6687 | 6731 | 6774 | 6818 | 6862 | 6906 | 44 |
| 12 | 3 | 6949 | 6993 | 7037 | 7080 | 7124 | 7168 | 7212 | 7255 | 7299 | 7343 | 44 |
| 18 | 4 | 7386 | 7430 | 7474 | 7517 | 7561 | 7605 | 7648 | 7692 | 7736 | 7779 | 44 |
| 22 | 5 | 7823 | 7867 | 7910 | 7954 | 7998 | 8041 | 8085 | 8129 | 8172 | 8216 | 44 |
| 26 | 6 | 8259 | 8303 | 8347 | 8390 | 8434 | 8477 | 8521 | 8564 | 8608 | 8652 | 44 |
| 30 | 7 | 8695 | 8739 | 8782 | 8826 | 8869 | 8913 | 8956 | 9000 | 9043 | 9087 | 44 |
| 35 | 8 | 9131 | 9174 | 9218 | 9261 | 9305 | 9348 | 9392 | 9435 | 9479 | 9522 | 44 |
| 40 | 9 | 9565 | 9609 | 9652 | 9696 | 9739 | 9783 | 9826 | 9870 | 9913 | 9957 | 43 |

A TABLE OF SQUARES, CUBES, AND ROOTS. 397

| No. | Square. | Cube. | Sq. Root. | CubeRoot | No. | Square. | Cube. | Sq. Root. | CubeRoot |
|-----|---------|--------|-----------|----------|-----|---------|---------|------------|----------|
| 1 | 1 | 1 | 1.000000 | 1.000000 | 64 | 4096 | 262144 | 8.000000 | 4.000000 |
| 2 | 4 | 8 | 1.4142136 | 1.259921 | 65 | 4225 | 274625 | 8.0622577 | 4.020726 |
| 3 | 9 | 27 | 1.7320508 | 1.442250 | 66 | 4356 | 287496 | 8.1210284 | 4.041210 |
| 4 | 16 | 64 | 2.0000000 | 1.587401 | 67 | 4489 | 300763 | 8.1855328 | 4.061548 |
| 5 | 25 | 125 | 2.2360680 | 1.709976 | 68 | 4624 | 314432 | 8.2462113 | 4.081656 |
| 6 | 36 | 216 | 2.4494897 | 1.817121 | 69 | 4761 | 328509 | 8.3062220 | 4.101566 |
| 7 | 49 | 343 | 2.6457513 | 1.912031 | 70 | 4900 | 343000 | 8.3666603 | 4.121285 |
| 8 | 64 | 512 | 2.8284271 | 2.000000 | 71 | 5041 | 357911 | 8.4264198 | 4.140818 |
| 9 | 81 | 729 | 3.0000000 | 2.080084 | 72 | 5184 | 373248 | 8.4865214 | 4.160168 |
| 10 | 100 | 1000 | 3.1622777 | 2.154435 | 73 | 5329 | 389017 | 8.5469937 | 4.179329 |
| 11 | 121 | 1331 | 3.3166248 | 2.233980 | 74 | 5476 | 405224 | 8.6078253 | 4.198336 |
| 12 | 144 | 1728 | 3.4641016 | 2.318328 | 75 | 5625 | 421875 | 8.6690250 | 4.217103 |
| 13 | 169 | 2197 | 3.6055513 | 2.407335 | 76 | 5776 | 438976 | 8.7305915 | 4.235734 |
| 14 | 196 | 2744 | 3.7416574 | 2.441012 | 77 | 5929 | 456523 | 8.7925414 | 4.254221 |
| 15 | 225 | 3375 | 3.8729832 | 2.466212 | 78 | 6084 | 474528 | 8.8548769 | 4.272569 |
| 16 | 256 | 4096 | 4.0000000 | 2.549842 | 79 | 6241 | 493009 | 8.9186144 | 4.290841 |
| 17 | 289 | 4913 | 4.1231056 | 2.571282 | 80 | 6400 | 512000 | 8.9442719 | 4.308870 |
| 18 | 324 | 5832 | 4.2426407 | 2.620741 | 81 | 6561 | 531441 | 9.0000000 | 4.326749 |
| 19 | 361 | 6859 | 4.3588989 | 2.668402 | 82 | 6724 | 551368 | 9.0565351 | 4.344481 |
| 20 | 400 | 8000 | 4.4721359 | 2.714418 | 83 | 6889 | 571787 | 9.1139336 | 4.362071 |
| 21 | 441 | 9261 | 4.5825757 | 2.758924 | 84 | 7056 | 592704 | 9.1631514 | 4.379519 |
| 22 | 484 | 10648 | 4.6901158 | 2.802039 | 85 | 7225 | 614125 | 9.2154545 | 4.396839 |
| 23 | 529 | 12167 | 4.7953315 | 2.843867 | 86 | 7396 | 636056 | 9.2708485 | 4.414005 |
| 24 | 576 | 13824 | 4.8989795 | 2.884499 | 87 | 7569 | 658503 | 9.3293391 | 4.431047 |
| 25 | 625 | 15625 | 5.0000000 | 2.924018 | 88 | 7744 | 681472 | 9.3908315 | 4.447960 |
| 26 | 676 | 17576 | 5.0990195 | 2.962469 | 89 | 7921 | 704969 | 9.4553281 | 4.464745 |
| 27 | 729 | 19683 | 5.1961524 | 3.000000 | 90 | 8100 | 729000 | 9.5228320 | 4.481405 |
| 28 | 784 | 21952 | 5.2915926 | 3.035589 | 91 | 8281 | 753571 | 9.5933920 | 4.497941 |
| 29 | 841 | 24389 | 5.3851618 | 3.072317 | 92 | 8464 | 778688 | 9.6670650 | 4.514357 |
| 30 | 900 | 27000 | 5.4772556 | 3.107232 | 93 | 8649 | 804357 | 9.7438695 | 4.530655 |
| 31 | 961 | 29791 | 5.5677644 | 3.141381 | 94 | 8836 | 830584 | 9.8238337 | 4.546836 |
| 32 | 1024 | 32768 | 5.6568542 | 3.174802 | 95 | 9025 | 857375 | 9.9069793 | 4.562903 |
| 33 | 1089 | 35937 | 5.7446626 | 3.207534 | 96 | 9216 | 884736 | 9.9933030 | 4.578857 |
| 34 | 1156 | 39304 | 5.8309519 | 3.239612 | 97 | 9409 | 912673 | 9.8858878 | 4.594701 |
| 35 | 1225 | 42875 | 5.9160798 | 3.271066 | 98 | 9604 | 941192 | 9.9828490 | 4.610426 |
| 36 | 1296 | 46656 | 6.0000000 | 3.301927 | 99 | 9801 | 970299 | 9.9849744 | 4.626065 |
| 37 | 1369 | 50653 | 6.0827625 | 3.332222 | 100 | 10000 | 1000000 | 10.0000000 | 4.641589 |
| 38 | 1444 | 54872 | 6.1644140 | 3.361975 | 101 | 10201 | 1030301 | 10.0498756 | 4.657010 |
| 39 | 1521 | 59319 | 6.2449980 | 3.391211 | 102 | 10404 | 1061208 | 10.0999049 | 4.672329 |
| 40 | 1600 | 64000 | 6.3245553 | 3.419952 | 103 | 10609 | 1092727 | 10.1489166 | 4.687548 |
| 41 | 1681 | 68921 | 6.4031242 | 3.448217 | 104 | 10816 | 1124864 | 10.1980030 | 4.702669 |
| 42 | 1764 | 74088 | 6.4807495 | 3.476027 | 105 | 11025 | 1157625 | 10.2463508 | 4.717694 |
| 43 | 1849 | 79507 | 6.5574385 | 3.503386 | 106 | 11236 | 1191016 | 10.2949631 | 4.732624 |
| 44 | 1936 | 85184 | 6.6332406 | 3.530348 | 107 | 11449 | 1225043 | 10.3438494 | 4.747459 |
| 45 | 2025 | 91125 | 6.7082039 | 3.556893 | 108 | 11664 | 1259712 | 10.3929208 | 4.762203 |
| 46 | 2116 | 97336 | 6.7823300 | 3.583048 | 109 | 11881 | 1295029 | 10.4422695 | 4.776856 |
| 47 | 2209 | 103823 | 6.8556516 | 3.608826 | 110 | 12100 | 1331000 | 10.4918985 | 4.791420 |
| 48 | 2304 | 110592 | 6.9282032 | 3.634241 | 111 | 12321 | 1367631 | 10.5418053 | 4.805886 |
| 49 | 2401 | 117649 | 7.0000000 | 3.659306 | 112 | 12544 | 1404928 | 10.5919952 | 4.820284 |
| 50 | 2500 | 125000 | 7.0710678 | 3.684031 | 113 | 12769 | 1442897 | 10.6301458 | 4.834588 |
| 51 | 2601 | 132651 | 7.1414284 | 3.708430 | 114 | 12996 | 1481544 | 10.6770753 | 4.848808 |
| 52 | 2704 | 140544 | 7.2111026 | 3.732511 | 115 | 13225 | 1520875 | 10.7238453 | 4.862944 |
| 53 | 2809 | 148677 | 7.2801009 | 3.756286 | 116 | 13456 | 1560896 | 10.7704226 | 4.877009 |
| 54 | 2916 | 157064 | 7.3484492 | 3.779763 | 117 | 13689 | 1601613 | 10.8168328 | 4.890973 |
| 55 | 3025 | 165705 | 7.4161983 | 3.802954 | 118 | 13924 | 1643032 | 10.8630780 | 4.904868 |
| 56 | 3136 | 174608 | 7.4833148 | 3.825862 | 119 | 14161 | 1685159 | 10.9091721 | 4.918685 |
| 57 | 3249 | 183873 | 7.5498314 | 3.848501 | 120 | 14400 | 1728000 | 10.9544512 | 4.932424 |
| 58 | 3364 | 193512 | 7.6157731 | 3.870877 | 121 | 14641 | 1771561 | 11.0000000 | 4.946088 |
| 59 | 3481 | 203529 | 7.6811453 | 3.892996 | 122 | 14884 | 1816864 | 11.0453610 | 4.959675 |
| 60 | 3600 | 214000 | 7.7459667 | 3.914867 | 123 | 15129 | 1863897 | 11.0905365 | 4.973190 |
| 61 | 3721 | 224941 | 7.8102437 | 3.936549 | 124 | 15376 | 1912672 | 11.1355287 | 4.986631 |
| 62 | 3844 | 236352 | 7.8740079 | 3.957932 | 125 | 15625 | 1963195 | 11.1803369 | 5.000000 |
| 63 | 3969 | 250047 | 7.9372539 | 3.979057 | 126 | 15876 | 2015476 | 11.2249722 | 5.013228 |

| No. | Square. | Cube. | Sq. Root. | CubeRoot | No. | Square. | Cube. | Sq. Root. | CubeRoot |
|-----|---------|---------|------------|----------|-----|---------|----------|------------|-----------|
| 127 | 16129 | 2048339 | 11-2694277 | 5-026526 | 190 | 36100 | 6850000 | 13-7840488 | 5-748807 |
| 128 | 16384 | 2097152 | 11-3117085 | 5-030684 | 191 | 36481 | 6967871 | 13-8202750 | 5-7588065 |
| 129 | 16641 | 2146689 | 11-3578197 | 5-035274 | 192 | 36864 | 7077888 | 13-8564095 | 5-7688098 |
| 130 | 16900 | 2197000 | 11-4075743 | 5-040297 | 193 | 37249 | 7189671 | 13-8924440 | 5-7788086 |
| 131 | 17161 | 2248101 | 11-4455241 | 5-045756 | 194 | 37636 | 7303884 | 13-9283888 | 5-7888089 |
| 132 | 17424 | 2299968 | 11-4891253 | 5-051643 | 195 | 38025 | 7418755 | 13-9642400 | 5-7988089 |
| 133 | 17689 | 2352637 | 11-5325262 | 5-104469 | 196 | 38416 | 7535336 | 14-0000000 | 5-8087866 |
| 134 | 17956 | 2406104 | 11-5758399 | 5-117230 | 197 | 38809 | 7653674 | 14-0356688 | 5-8187424 |
| 135 | 18225 | 2460375 | 11-6189500 | 5-129928 | 198 | 39204 | 7773801 | 14-0712474 | 5-8286844 |
| 136 | 18496 | 2515456 | 11-6619608 | 5-142563 | 199 | 39601 | 7895669 | 14-1067390 | 5-8386476 |
| 137 | 18769 | 2571353 | 11-7048799 | 5-155137 | 200 | 40000 | 8000000 | 14-1421356 | 5-8486356 |
| 138 | 19044 | 2628072 | 11-7475344 | 5-167649 | 201 | 40401 | 8120611 | 14-1774469 | 5-8587596 |
| 139 | 19321 | 2685619 | 11-7899254 | 5-180100 | 202 | 40804 | 8242408 | 14-2126704 | 5-8689464 |
| 140 | 19600 | 2744000 | 11-8321576 | 5-192499 | 203 | 41209 | 8365427 | 14-2477068 | 5-8791939 |
| 141 | 19881 | 2803221 | 11-8743321 | 5-204848 | 204 | 41616 | 8489664 | 14-2826579 | 5-8895065 |
| 142 | 20164 | 2863288 | 11-9163533 | 5-217147 | 205 | 42025 | 8615125 | 14-3175224 | 5-8998868 |
| 143 | 20449 | 2924207 | 11-9582267 | 5-229397 | 206 | 42436 | 8741816 | 14-3522901 | 5-9103241 |
| 144 | 20736 | 2985984 | 12-0000000 | 5-241598 | 207 | 42849 | 8869841 | 14-3869616 | 5-9208189 |
| 145 | 21025 | 3048625 | 12-0415916 | 5-253758 | 208 | 43264 | 8999112 | 14-4222254 | 5-9313733 |
| 146 | 21316 | 3112136 | 12-0830160 | 5-265877 | 209 | 43681 | 9129629 | 14-4576823 | 5-9419873 |
| 147 | 21609 | 3176523 | 12-1243537 | 5-277952 | 210 | 44100 | 9261400 | 14-4933437 | 5-9526619 |
| 148 | 21904 | 3241792 | 12-1655251 | 5-290072 | 211 | 44521 | 9394531 | 14-5292080 | 5-9633971 |
| 149 | 22201 | 3307949 | 12-2065536 | 5-302137 | 212 | 44944 | 9528928 | 14-5652800 | 5-9741939 |
| 150 | 22500 | 3375000 | 12-2474487 | 5-314158 | 213 | 45369 | 9664587 | 14-6015615 | 5-9850524 |
| 151 | 22801 | 3442961 | 12-2882056 | 5-326124 | 214 | 45796 | 9801514 | 14-6380478 | 5-9959727 |
| 152 | 23104 | 3511838 | 12-3288280 | 5-338035 | 215 | 46225 | 9939715 | 14-6747400 | 5-9999999 |
| 153 | 23409 | 3581637 | 12-3693169 | 5-349891 | 216 | 46656 | 10079288 | 14-7116585 | 6-0000000 |
| 154 | 23716 | 3652364 | 12-4096736 | 5-361692 | 217 | 47089 | 10220331 | 14-7487940 | 6-0002244 |
| 155 | 24025 | 3724025 | 12-4499096 | 5-373438 | 218 | 47524 | 10362852 | 14-7861481 | 6-0004684 |
| 156 | 24336 | 3796616 | 12-4899960 | 5-385129 | 219 | 47961 | 10506849 | 14-8237216 | 6-0007260 |
| 157 | 24649 | 3870133 | 12-5299641 | 5-396765 | 220 | 48400 | 10652320 | 14-8615250 | 6-0009881 |
| 158 | 24964 | 3944688 | 12-5698051 | 5-408346 | 221 | 48841 | 10799271 | 14-8995579 | 6-0012544 |
| 159 | 25281 | 4020289 | 12-6095202 | 5-420071 | 222 | 49284 | 10947704 | 14-8996644 | 6-0015260 |
| 160 | 25600 | 4096900 | 12-6491106 | 5-431841 | 223 | 49729 | 11097625 | 14-9331845 | 6-0018026 |
| 161 | 25921 | 4173521 | 12-6885777 | 5-443656 | 224 | 50176 | 11249044 | 14-9666265 | 6-0020842 |
| 162 | 26244 | 4251152 | 12-7279221 | 5-455516 | 225 | 50625 | 11401961 | 14-9999900 | 6-0023707 |
| 163 | 26569 | 4329793 | 12-7671453 | 5-467421 | 226 | 51076 | 11555484 | 15-0332843 | 6-0026621 |
| 164 | 26896 | 4409444 | 12-8062483 | 5-479370 | 227 | 51529 | 11709623 | 15-0665969 | 6-0029584 |
| 165 | 27225 | 4490105 | 12-8452326 | 5-491364 | 228 | 51984 | 11874376 | 15-1000299 | 6-0032596 |
| 166 | 27556 | 4571786 | 12-8841987 | 5-495879 | 229 | 52441 | 12040741 | 15-1334834 | 6-0035657 |
| 167 | 27889 | 4654487 | 12-9231480 | 5-500829 | 230 | 52900 | 12208720 | 15-1670570 | 6-0038769 |
| 168 | 28224 | 4738208 | 12-9620814 | 5-515818 | 231 | 53361 | 12378321 | 15-1806424 | 6-0041922 |
| 169 | 28561 | 4822949 | 12-9999960 | 5-520757 | 232 | 53824 | 12549544 | 15-2143402 | 6-0045126 |
| 170 | 28900 | 4908700 | 13-0368908 | 5-525746 | 233 | 54289 | 12722389 | 15-2481515 | 6-0048379 |
| 171 | 29241 | 5000481 | 13-0737658 | 5-530785 | 234 | 54756 | 12896856 | 15-2820768 | 6-0051681 |
| 172 | 29584 | 5088284 | 13-1106210 | 5-535874 | 235 | 55225 | 13072945 | 15-3161169 | 6-0055033 |
| 173 | 29929 | 5177117 | 13-1474564 | 5-541013 | 236 | 55696 | 13250664 | 15-3502720 | 6-0058444 |
| 174 | 30276 | 5266980 | 13-1842720 | 5-546202 | 237 | 56169 | 13429911 | 15-3845431 | 6-0061915 |
| 175 | 30625 | 5357875 | 13-2210686 | 5-551441 | 238 | 56644 | 13610684 | 15-4189302 | 6-0065446 |
| 176 | 30976 | 5449804 | 13-2578464 | 5-556730 | 239 | 57121 | 13793081 | 15-4534343 | 6-0069037 |
| 177 | 31329 | 5542767 | 13-2946054 | 5-562069 | 240 | 57600 | 13977092 | 15-4880554 | 6-0072688 |
| 178 | 31684 | 5636764 | 13-3313464 | 5-567458 | 241 | 58081 | 14162717 | 15-5227935 | 6-0076409 |
| 179 | 32041 | 5731799 | 13-3680694 | 5-572897 | 242 | 58564 | 14349956 | 15-5576486 | 6-0080190 |
| 180 | 32400 | 5827872 | 13-4047744 | 5-578386 | 243 | 59049 | 14538809 | 15-5926207 | 6-0084031 |
| 181 | 32761 | 5924985 | 13-4414614 | 5-583925 | 244 | 59536 | 14729276 | 15-6277098 | 6-0087932 |
| 182 | 33124 | 6023138 | 13-4781304 | 5-589514 | 245 | 60025 | 14921357 | 15-6629159 | 6-0091893 |
| 183 | 33489 | 6122341 | 13-5247794 | 5-595153 | 246 | 60516 | 15115052 | 15-6982390 | 6-0095914 |
| 184 | 33856 | 6222604 | 13-5614084 | 5-587734 | 247 | 61009 | 15310361 | 15-7336791 | 6-0099995 |
| 185 | 34225 | 6323925 | 13-6000174 | 5-593373 | 248 | 61504 | 15507284 | 15-7692362 | 6-0104136 |
| 186 | 34596 | 6426306 | 13-6386164 | 5-599072 | 249 | 62001 | 15705821 | 15-8049113 | 6-0108337 |
| 187 | 34969 | 6529747 | 13-6771954 | 5-604821 | 250 | 62500 | 15905972 | 15-8407034 | 6-0112598 |
| 188 | 35344 | 6634248 | 13-7157544 | 5-610620 | 251 | 63001 | 16107737 | 15-8766135 | 6-0116919 |
| 189 | 35721 | 6739809 | 13-7542934 | 5-616469 | 252 | 63504 | 16311116 | 15-9126416 | 6-0121290 |

SQUARES, CUBES, AND ROOTS.

399

| No. | Square. | Cube. | Sq. Root. | CubeRoot | No. | Square. | Cube. | Sq. Root. | CubeRoot |
|-----|---------|----------|------------|----------|-----|---------|----------|------------|-----------|
| 203 | 64009 | 16194277 | 15-9059737 | 6-324704 | 316 | 99856 | 3154466 | 17-7763888 | 6-811284 |
| 204 | 64516 | 16387064 | 15-9373775 | 6-339026 | 317 | 100189 | 31855013 | 17-8044098 | 6-818462 |
| 205 | 65023 | 16581375 | 15-9687194 | 6-354326 | 318 | 101124 | 32157432 | 17-8325465 | 6-825640 |
| 206 | 65536 | 16777216 | 16-0000000 | 6-369604 | 319 | 101764 | 32460759 | 17-8605511 | 6-8327771 |
| 207 | 66049 | 16974583 | 16-0312195 | 6-384861 | 320 | 102400 | 32766000 | 17-8885438 | 6-839904 |
| 208 | 66564 | 17173512 | 16-0623784 | 6-366095 | 321 | 102941 | 33073161 | 17-9164720 | 6-847021 |
| 209 | 67081 | 17373979 | 16-0934769 | 6-374311 | 322 | 103584 | 33382248 | 17-9443584 | 6-8541214 |
| 210 | 67600 | 17576000 | 16-1245153 | 6-382594 | 323 | 104239 | 33693267 | 17-9722008 | 6-8612122 |
| 211 | 68121 | 17779581 | 16-1554944 | 6-390976 | 324 | 104896 | 34006224 | 18-0000000 | 6-8682825 |
| 212 | 68644 | 17984728 | 16-1864441 | 6-399466 | 325 | 105656 | 34321215 | 18-0277564 | 6-8753444 |
| 213 | 69169 | 18191447 | 16-2172747 | 6-408065 | 326 | 106420 | 34638246 | 18-0554701 | 6-8823988 |
| 214 | 69696 | 18399744 | 16-2480768 | 6-416068 | 327 | 106929 | 34957383 | 18-0831413 | 6-8894419 |
| 215 | 70225 | 18609625 | 16-2788206 | 6-424315 | 328 | 107584 | 35278552 | 18-1107705 | 6-8964845 |
| 216 | 70756 | 18821096 | 16-3095964 | 6-432828 | 329 | 108244 | 35601789 | 18-1383574 | 6-9035266 |
| 217 | 71289 | 19034163 | 16-3404346 | 6-441527 | 330 | 108900 | 35927100 | 18-1659021 | 6-9105683 |
| 218 | 71824 | 19248832 | 16-3707055 | 6-447305 | 331 | 109561 | 36254491 | 18-1934954 | 6-9176106 |
| 219 | 72361 | 19465109 | 16-4012195 | 6-453315 | 332 | 110224 | 36583968 | 18-2210472 | 6-9246525 |
| 220 | 72900 | 19683000 | 16-4316767 | 6-463304 | 333 | 110899 | 36916537 | 18-2485276 | 6-9316941 |
| 221 | 73441 | 19902511 | 16-4620776 | 6-471274 | 334 | 111576 | 37251204 | 18-2759260 | 6-9387354 |
| 222 | 73984 | 20123648 | 16-4924225 | 6-479224 | 335 | 112256 | 37587975 | 18-3032905 | 6-9457765 |
| 223 | 74529 | 20346417 | 16-5227116 | 6-487154 | 336 | 112936 | 37926856 | 18-3306288 | 6-9528173 |
| 224 | 75076 | 20570824 | 16-5529454 | 6-495065 | 337 | 113619 | 38267853 | 18-3579598 | 6-9598579 |
| 225 | 75625 | 20796875 | 16-5831240 | 6-502956 | 338 | 114244 | 38611072 | 18-3852763 | 6-9668984 |
| 226 | 76176 | 21024576 | 16-6132477 | 6-510829 | 339 | 114921 | 38956519 | 18-4125826 | 6-9739388 |
| 227 | 76729 | 21253933 | 16-6433170 | 6-518684 | 340 | 115600 | 39304100 | 18-4398889 | 6-9809791 |
| 228 | 77284 | 21484952 | 16-6733320 | 6-526519 | 341 | 116281 | 39653821 | 18-4671954 | 6-9880193 |
| 229 | 77841 | 21717639 | 16-7032931 | 6-534335 | 342 | 116964 | 40005688 | 18-4945020 | 6-9950594 |
| 230 | 78400 | 21952000 | 16-7332005 | 6-542133 | 343 | 117649 | 40359707 | 18-5218092 | 7-0000000 |
| 231 | 78961 | 22188041 | 16-7630546 | 6-549912 | 344 | 118336 | 40716884 | 18-5491230 | 7-0069676 |
| 232 | 79524 | 22425768 | 16-7928556 | 6-557672 | 345 | 119025 | 41076225 | 18-5764379 | 7-0139359 |
| 233 | 80089 | 22665187 | 16-8226038 | 6-565415 | 346 | 119716 | 41437736 | 18-6037552 | 7-0209049 |
| 234 | 80656 | 22906304 | 16-8522995 | 6-573139 | 347 | 120409 | 41801423 | 18-6320960 | 7-0278746 |
| 235 | 81225 | 23149125 | 16-8819430 | 6-580844 | 348 | 121104 | 42167292 | 18-6604704 | 7-0348449 |
| 236 | 81796 | 23393656 | 16-9115345 | 6-588532 | 349 | 121801 | 42535339 | 18-6888784 | 7-0418158 |
| 237 | 82369 | 23639903 | 16-9410743 | 6-596202 | 350 | 122500 | 42905570 | 18-7173200 | 7-0487873 |
| 238 | 82944 | 23887872 | 16-9705627 | 6-603854 | 351 | 123201 | 43277991 | 18-7458040 | 7-0557594 |
| 239 | 83521 | 24137569 | 17-0000000 | 6-611489 | 352 | 123904 | 43652608 | 18-7743096 | 7-0627321 |
| 240 | 84100 | 24388984 | 17-0294864 | 6-619106 | 353 | 124609 | 44029427 | 18-7828292 | 7-0697054 |
| 241 | 84681 | 24642117 | 17-0589721 | 6-626705 | 354 | 125316 | 44408456 | 18-8113877 | 7-0766793 |
| 242 | 85264 | 24897088 | 17-0884575 | 6-634287 | 355 | 126025 | 44789691 | 18-8400000 | 7-0836538 |
| 243 | 85849 | 25153875 | 17-1179428 | 6-641852 | 356 | 126736 | 45173136 | 18-8686663 | 7-0906289 |
| 244 | 86436 | 25412484 | 17-1474282 | 6-649399 | 357 | 127449 | 45558895 | 18-8973856 | 7-0976046 |
| 245 | 87025 | 25672913 | 17-1769138 | 6-656930 | 358 | 128164 | 45946974 | 18-9261588 | 7-1045809 |
| 246 | 87616 | 25935264 | 17-2064005 | 6-664444 | 359 | 128881 | 46337389 | 18-9549859 | 7-1115578 |
| 247 | 88209 | 26199537 | 17-2358879 | 6-671940 | 360 | 129600 | 46730104 | 18-9838670 | 7-1185353 |
| 248 | 88804 | 26465832 | 17-2653762 | 6-679420 | 361 | 130321 | 47125123 | 19-0128020 | 7-1255134 |
| 249 | 89401 | 26734159 | 17-2948655 | 6-686882 | 362 | 131044 | 47522452 | 19-0417916 | 7-1324921 |
| 250 | 90000 | 27004500 | 17-3243558 | 6-694326 | 363 | 131769 | 47922091 | 19-0708356 | 7-1394814 |
| 251 | 90601 | 27276961 | 17-3538472 | 6-701755 | 364 | 132496 | 48324046 | 19-0999340 | 7-1464813 |
| 252 | 91204 | 27551536 | 17-3833397 | 6-709173 | 365 | 133225 | 48728313 | 19-1290868 | 7-1534918 |
| 253 | 91809 | 27828225 | 17-4028332 | 6-716579 | 366 | 133956 | 49134898 | 19-1582930 | 7-1605029 |
| 254 | 92416 | 28106928 | 17-4223277 | 6-723974 | 367 | 134689 | 49543807 | 19-1875524 | 7-1675146 |
| 255 | 93025 | 28388645 | 17-4418232 | 6-731358 | 368 | 135424 | 49955036 | 19-2168652 | 7-1745269 |
| 256 | 93636 | 28672376 | 17-4613197 | 6-738731 | 369 | 136161 | 50368581 | 19-2462314 | 7-1815398 |
| 257 | 94249 | 28959121 | 17-5211155 | 6-746094 | 370 | 136900 | 50784448 | 19-2756500 | 7-1885533 |
| 258 | 94864 | 29248880 | 17-5409114 | 6-753447 | 371 | 137641 | 51202633 | 19-3051202 | 7-1955674 |
| 259 | 95481 | 29541661 | 17-5707073 | 6-760791 | 372 | 138384 | 51623142 | 19-3346420 | 7-2025921 |
| 260 | 96100 | 29837464 | 17-6005032 | 6-768125 | 373 | 139129 | 52045971 | 19-3642154 | 7-2096174 |
| 261 | 96721 | 30136289 | 17-6303001 | 6-775459 | 374 | 139876 | 52471126 | 19-3938404 | 7-2166433 |
| 262 | 97344 | 30438136 | 17-6600970 | 6-782793 | 375 | 140625 | 52898603 | 19-4235168 | 7-2236698 |
| 263 | 97969 | 30743005 | 17-6898939 | 6-790127 | 376 | 141376 | 53328408 | 19-4532446 | 7-2306969 |
| 264 | 98596 | 31051896 | 17-7196908 | 6-797461 | 377 | 142129 | 53760547 | 19-4830238 | 7-2377246 |
| 265 | 99225 | 31364809 | 17-7494877 | 6-804795 | 378 | 142884 | 54195026 | 19-5128544 | 7-2447529 |

| No. | Square. | Cube. | Sq. Root. | CubeRoot | No. | Square. | Cube. | Sq. Root. | CubeRoot |
|-----|---------|----------|------------|----------|-----|---------|-----------|------------|----------|
| 379 | 143641 | 54420020 | 19.4679223 | 7.226797 | 442 | 195364 | 86350888 | 21.0237960 | 7.617412 |
| 380 | 144400 | 54872000 | 19.495587 | 7.243156 | 443 | 196249 | 86938307 | 21.0475562 | 7.623152 |
| 381 | 145161 | 55363341 | 19.5102213 | 7.249504 | 444 | 197156 | 87528384 | 21.0713075 | 7.628884 |
| 382 | 145924 | 55742684 | 19.5148293 | 7.255844 | 445 | 198025 | 88121125 | 21.0950231 | 7.634607 |
| 383 | 146689 | 56181857 | 19.5708588 | 7.262166 | 446 | 198946 | 88716536 | 21.1187121 | 7.640321 |
| 384 | 147456 | 56629104 | 19.5959179 | 7.268482 | 447 | 199849 | 89314623 | 21.1423735 | 7.646027 |
| 385 | 148225 | 57066625 | 19.6214169 | 7.274786 | 448 | 200744 | 89915392 | 21.1660105 | 7.651725 |
| 386 | 148996 | 57512456 | 19.6468827 | 7.281079 | 449 | 201641 | 90518849 | 21.1896201 | 7.657414 |
| 387 | 149769 | 57966603 | 19.6723156 | 7.287362 | 450 | 202540 | 91125000 | 21.2132034 | 7.663094 |
| 388 | 150544 | 58411072 | 19.6977156 | 7.293633 | 451 | 203441 | 91733851 | 21.2367666 | 7.668766 |
| 389 | 151321 | 58864869 | 19.7230829 | 7.299894 | 452 | 204344 | 92345408 | 21.2602916 | 7.674430 |
| 390 | 152100 | 59319000 | 19.7484177 | 7.306143 | 453 | 205249 | 92959677 | 21.2837797 | 7.680088 |
| 391 | 152881 | 59773489 | 19.7737189 | 7.312383 | 454 | 206156 | 93576664 | 21.3072758 | 7.685733 |
| 392 | 153664 | 60228288 | 19.7989849 | 7.318611 | 455 | 207065 | 94196375 | 21.3307758 | 7.691372 |
| 393 | 154449 | 60683457 | 19.8242276 | 7.324828 | 456 | 207976 | 94818816 | 21.3542765 | 7.697007 |
| 394 | 155236 | 61138924 | 19.8494432 | 7.331037 | 457 | 208889 | 95443999 | 21.3777733 | 7.702629 |
| 395 | 156025 | 61602875 | 19.8746369 | 7.337234 | 458 | 209804 | 96071912 | 21.4008346 | 7.708239 |
| 396 | 156816 | 62069336 | 19.8997487 | 7.343419 | 459 | 210721 | 96702579 | 21.4242853 | 7.713845 |
| 397 | 157609 | 62539373 | 19.9248588 | 7.349597 | 460 | 211640 | 97335000 | 21.4471676 | 7.719442 |
| 398 | 158404 | 63012992 | 19.9499673 | 7.355762 | 461 | 212561 | 97970281 | 21.4701166 | 7.725032 |
| 399 | 159201 | 63491199 | 19.9749844 | 7.361918 | 462 | 213484 | 98608424 | 21.4931853 | 7.730614 |
| 400 | 160000 | 64000000 | 20.0000000 | 7.368063 | 463 | 214409 | 99249529 | 21.5163438 | 7.736188 |
| 401 | 160801 | 64481201 | 20.0249844 | 7.374198 | 464 | 215336 | 99893684 | 21.5395992 | 7.741753 |
| 402 | 161604 | 64964808 | 20.0499377 | 7.380322 | 465 | 216265 | 100540625 | 21.5628587 | 7.747311 |
| 403 | 162409 | 65450827 | 20.0748599 | 7.386437 | 466 | 217196 | 101194496 | 21.5861631 | 7.752861 |
| 404 | 163216 | 65939264 | 20.0997512 | 7.392543 | 467 | 218129 | 101854753 | 21.6095128 | 7.758402 |
| 405 | 164025 | 66430125 | 20.1246118 | 7.398646 | 468 | 219064 | 102521532 | 21.6328307 | 7.763936 |
| 406 | 164836 | 66923416 | 20.1494417 | 7.404729 | 469 | 219991 | 103194919 | 21.6561476 | 7.769462 |
| 407 | 165649 | 67419143 | 20.1742410 | 7.410795 | 470 | 220920 | 103874900 | 21.6794834 | 7.774980 |
| 408 | 166464 | 67917312 | 20.1990099 | 7.416859 | 471 | 221851 | 104560581 | 21.7028334 | 7.780490 |
| 409 | 167281 | 68417929 | 20.2237481 | 7.422914 | 472 | 222784 | 105252068 | 21.7261961 | 7.785993 |
| 410 | 168100 | 68921000 | 20.2484567 | 7.428969 | 473 | 223719 | 105949369 | 21.7495827 | 7.791487 |
| 411 | 168921 | 69426531 | 20.2731349 | 7.435014 | 474 | 224656 | 106652484 | 21.7729441 | 7.796974 |
| 412 | 169744 | 69934528 | 20.2977831 | 7.441049 | 475 | 225595 | 107362409 | 21.7962854 | 7.802454 |
| 413 | 170569 | 70444997 | 20.3224014 | 7.447074 | 476 | 226536 | 108079176 | 21.8196122 | 7.807925 |
| 414 | 171396 | 70957944 | 20.3469899 | 7.453090 | 477 | 227479 | 108802799 | 21.8468327 | 7.813389 |
| 415 | 172225 | 71473375 | 20.3715488 | 7.459096 | 478 | 228424 | 109533284 | 21.8632111 | 7.818846 |
| 416 | 173056 | 71991296 | 20.3960781 | 7.465092 | 479 | 229371 | 110270529 | 21.8896686 | 7.824294 |
| 417 | 173889 | 72511713 | 20.4205779 | 7.471079 | 480 | 230320 | 111014540 | 21.9161923 | 7.829735 |
| 418 | 174724 | 73034632 | 20.4450483 | 7.477056 | 481 | 231271 | 111765341 | 21.9327122 | 7.835169 |
| 419 | 175561 | 73560059 | 20.4694895 | 7.483024 | 482 | 232224 | 112522944 | 21.9492184 | 7.840585 |
| 420 | 176400 | 74088000 | 20.4939015 | 7.488982 | 483 | 233179 | 113287359 | 21.9727260 | 7.846013 |
| 421 | 177241 | 74618461 | 20.5182845 | 7.494931 | 484 | 234136 | 114058592 | 22.0000000 | 7.851424 |
| 422 | 178084 | 75151448 | 20.5426386 | 7.500871 | 485 | 235095 | 114836753 | 22.0272155 | 7.856828 |
| 423 | 178929 | 75686967 | 20.5669638 | 7.506804 | 486 | 236056 | 115621944 | 22.0543767 | 7.862224 |
| 424 | 179776 | 76225024 | 20.5912603 | 7.512721 | 487 | 237019 | 116414179 | 22.0814856 | 7.867613 |
| 425 | 180625 | 76765625 | 20.6155281 | 7.518634 | 488 | 238084 | 117213464 | 22.0972220 | 7.872994 |
| 426 | 181476 | 77308776 | 20.6397674 | 7.524535 | 489 | 239151 | 118019809 | 22.1133444 | 7.878373 |
| 427 | 182329 | 77854483 | 20.6639783 | 7.530424 | 490 | 240220 | 118833320 | 22.1293466 | 7.883735 |
| 428 | 183184 | 78402752 | 20.6881609 | 7.536301 | 491 | 241291 | 119654009 | 22.1453498 | 7.889085 |
| 429 | 184041 | 78953589 | 20.7123152 | 7.542166 | 492 | 242364 | 120481868 | 22.1613730 | 7.894427 |
| 430 | 184900 | 79507000 | 20.7364414 | 7.548024 | 493 | 243449 | 121316909 | 22.2039403 | 7.899732 |
| 431 | 185761 | 80063091 | 20.7605395 | 7.553878 | 494 | 244536 | 122159244 | 22.2261108 | 7.905029 |
| 432 | 186624 | 80621868 | 20.7846197 | 7.559726 | 495 | 245625 | 123008975 | 22.2483955 | 7.910340 |
| 433 | 187489 | 81183337 | 20.8086820 | 7.565563 | 496 | 246716 | 123866104 | 22.2707167 | 7.915673 |
| 434 | 188356 | 81747504 | 20.8327267 | 7.571394 | 497 | 247809 | 124730739 | 22.2930768 | 7.921100 |
| 435 | 189225 | 82314475 | 20.8567536 | 7.577224 | 498 | 248904 | 125602880 | 22.3154763 | 7.926548 |
| 436 | 190096 | 82884256 | 20.8807630 | 7.583054 | 499 | 250001 | 126482529 | 22.3379161 | 7.932010 |
| 437 | 190969 | 83456853 | 20.9047450 | 7.588879 | 500 | 251100 | 127369680 | 22.3603978 | 7.937495 |
| 438 | 191844 | 84032264 | 20.9287195 | 7.594703 | 501 | 252201 | 128264339 | 22.3829263 | 7.942993 |
| 439 | 192721 | 84610499 | 20.9526868 | 7.600518 | 502 | 253304 | 129166504 | 22.4055023 | 7.948514 |
| 440 | 193600 | 85191660 | 20.9766470 | 7.606325 | 503 | 254409 | 129976289 | 22.4281261 | 7.954048 |
| 441 | 194481 | 85775749 | 21.0006000 | 7.612122 | 504 | 255516 | 130793696 | 22.4507994 | 7.959614 |

SQUARES, CUBES, AND ROOTS.

| No. | Square. | Cube. | Sq. Root. | CubeRoot | No. | Square. | Cube. | Sq. Root. | CubeRoot |
|-----|---------|-----------|------------|----------|-----|---------|-----------|------------|----------|
| 505 | 255025 | 128787625 | 22-4722051 | 7-963374 | 568 | 322624 | 183250432 | 23-832796 | 8-218635 |
| 506 | 256036 | 129554216 | 22-4944438 | 7-968627 | 569 | 323761 | 184220009 | 23-853729 | 8-226493 |
| 507 | 257049 | 130323843 | 22-5166605 | 7-973873 | 570 | 324900 | 185193000 | 23-8746728 | 8-291344 |
| 508 | 258064 | 131106512 | 22-5388553 | 7-979112 | 571 | 326041 | 186169411 | 23-8966063 | 8-296190 |
| 509 | 259081 | 131872225 | 22-5610293 | 7-984344 | 572 | 327184 | 187149248 | 23-9186215 | 8-301030 |
| 510 | 260100 | 132651000 | 22-5831796 | 7-989570 | 573 | 328329 | 188132517 | 23-9404184 | 8-305865 |
| 511 | 261121 | 133432831 | 22-6053091 | 7-994788 | 574 | 329476 | 189119224 | 23-9629271 | 8-310694 |
| 512 | 262144 | 134217728 | 22-6274170 | 8-000000 | 575 | 330625 | 190109375 | 23-9745576 | 8-315517 |
| 513 | 263169 | 135005697 | 22-6495033 | 8-005205 | 576 | 331776 | 191102976 | 24-0000000 | 8-320335 |
| 514 | 264196 | 135796744 | 22-6715681 | 8-010403 | 577 | 332929 | 192100033 | 24-0208243 | 8-325147 |
| 515 | 265225 | 136590875 | 22-6936134 | 8-015595 | 578 | 334084 | 193100552 | 24-0416306 | 8-329954 |
| 516 | 266256 | 137388096 | 22-7156391 | 8-020779 | 579 | 335241 | 194104539 | 24-0624188 | 8-334765 |
| 517 | 267289 | 138188413 | 22-7376340 | 8-025957 | 580 | 336400 | 195112000 | 24-0831892 | 8-339581 |
| 518 | 268324 | 138991832 | 22-7596134 | 8-031129 | 581 | 337561 | 196122941 | 24-1039416 | 8-344341 |
| 519 | 269361 | 139798369 | 22-7815715 | 8-036293 | 582 | 338724 | 197137368 | 24-1246732 | 8-349126 |
| 520 | 270400 | 140608000 | 22-8035085 | 8-041451 | 583 | 339889 | 198155287 | 24-1453829 | 8-353905 |
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SQUARES, CUBES, AND ROOTS.

403

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| 809 | 654481 | 529332661 | 28-4353670 | 9-318676 | 872 | 759486 | 663458440 | 29-7651011 | 9-558185 |
| 810 | 656100 | 531441000 | 28-4521204 | 9-322561 | 873 | 761227 | 665768359 | 29-7938312 | 9-562064 |
| 811 | 657721 | 533454411 | 28-4688728 | 9-326444 | 874 | 762970 | 668084328 | 29-8225633 | 9-565943 |
| 812 | 659344 | 535472824 | 28-4856242 | 9-330325 | 875 | 764715 | 670406347 | 29-8512974 | 9-569822 |
| 813 | 660969 | 537496243 | 28-5023746 | 9-334204 | 876 | 766462 | 672734416 | 29-8800335 | 9-573701 |
| 814 | 662596 | 539524668 | 28-5191240 | 9-338081 | 877 | 768211 | 675068535 | 29-9087716 | 9-577580 |
| 815 | 664225 | 541558103 | 28-5358724 | 9-341956 | 878 | 770000 | 677408704 | 29-9375117 | 9-581459 |
| 816 | 665856 | 543596546 | 28-5526198 | 9-345829 | 879 | 771791 | 679754923 | 29-9662538 | 9-585338 |
| 817 | 667489 | 545640000 | 28-5693662 | 9-349700 | 880 | 773584 | 682107192 | 29-9950000 | 9-589217 |
| 818 | 669124 | 547688463 | 28-5861116 | 9-353569 | 881 | 775379 | 684465511 | 29-1023791 | 9-593096 |
| 819 | 670761 | 549741936 | 28-6181700 | 9-357436 | 882 | 777174 | 686829880 | 29-1092682 | 9-596975 |

| No. | Square. | Cube. | Sq. Root. | CubeRoot | No. | Square. | Cube. | Sq. Root. | CubeRoot |
|-----|---------|-----------|------------|----------|------|---------|------------|------------|-----------|
| 883 | 779689 | 638456387 | 29-7163159 | 9-503716 | 942 | 887364 | 835986888 | 30-6920185 | 9-802804 |
| 884 | 781456 | 690907104 | 29-7321375 | 9-597337 | 943 | 889249 | 838561807 | 30-7083051 | 9-806271 |
| 885 | 783225 | 693154115 | 29-7489496 | 9-600955 | 944 | 891136 | 841232384 | 30-7245830 | 9-809736 |
| 886 | 784996 | 695506456 | 29-7657521 | 9-604570 | 945 | 893025 | 843908625 | 30-7408523 | 9-813199 |
| 887 | 786769 | 697964103 | 29-7825452 | 9-608182 | 946 | 894916 | 846590536 | 30-7571130 | 9-816659 |
| 888 | 788544 | 700227072 | 29-7993289 | 9-611791 | 947 | 896809 | 849278123 | 30-7733651 | 9-820117 |
| 889 | 790321 | 702595939 | 29-8161030 | 9-615398 | 948 | 898704 | 851971392 | 30-7896056 | 9-823572 |
| 890 | 792100 | 704969800 | 29-8328675 | 9-619002 | 949 | 900601 | 854670349 | 30-8058436 | 9-827025 |
| 891 | 793881 | 707343971 | 29-8496231 | 9-622603 | 950 | 902500 | 857375000 | 30-8220700 | 9-830476 |
| 892 | 795664 | 709722258 | 29-8663690 | 9-626201 | 951 | 904401 | 860085351 | 30-8382579 | 9-833924 |
| 893 | 797449 | 712112957 | 29-8831056 | 9-629787 | 952 | 906304 | 862801408 | 30-8544972 | 9-837369 |
| 894 | 799236 | 714516984 | 29-8998328 | 9-633389 | 953 | 908209 | 865523177 | 30-8706981 | 9-840813 |
| 895 | 801025 | 716917375 | 29-9165506 | 9-636981 | 954 | 910116 | 868250664 | 30-8869094 | 9-844254 |
| 896 | 802816 | 719323136 | 29-9332591 | 9-640569 | 955 | 912025 | 870983875 | 30-9030743 | 9-847692 |
| 897 | 804609 | 721734273 | 29-9499583 | 9-644154 | 956 | 913936 | 873722316 | 30-9192497 | 9-851128 |
| 898 | 806404 | 724150792 | 29-9666481 | 9-647737 | 957 | 915849 | 876467493 | 30-9354166 | 9-854562 |
| 899 | 808201 | 726572699 | 29-9833287 | 9-651317 | 958 | 917764 | 879217912 | 30-9515751 | 9-857993 |
| 900 | 810000 | 729000000 | 30-0000000 | 9-654894 | 959 | 919681 | 881974079 | 30-9677255 | 9-861422 |
| 901 | 811801 | 731431701 | 30-0166620 | 9-658468 | 960 | 921600 | 884736000 | 30-9838668 | 9-864848 |
| 902 | 813604 | 733878008 | 30-0333148 | 9-662049 | 961 | 923521 | 887503681 | 31-0000000 | 9-868272 |
| 903 | 815409 | 736339159 | 30-0499584 | 9-665609 | 962 | 925444 | 890277128 | 31-0161248 | 9-871694 |
| 904 | 817216 | 738816264 | 30-0665928 | 9-669176 | 963 | 927369 | 893056347 | 31-0322413 | 9-875113 |
| 905 | 819025 | 741318325 | 30-0832179 | 9-672740 | 964 | 929296 | 895841344 | 31-0483494 | 9-878520 |
| 906 | 820836 | 743836416 | 30-0998339 | 9-676302 | 965 | 931225 | 898632125 | 31-0644491 | 9-881945 |
| 907 | 822649 | 746360543 | 30-1164407 | 9-679860 | 966 | 933156 | 901428696 | 31-0805405 | 9-885357 |
| 908 | 824464 | 748890712 | 30-1330383 | 9-683416 | 967 | 935089 | 904231063 | 31-0966236 | 9-888767 |
| 909 | 826281 | 751026929 | 30-1496269 | 9-686970 | 968 | 937024 | 907033232 | 31-1126994 | 9-892175 |
| 910 | 828100 | 753371000 | 30-1662060 | 9-690521 | 969 | 938961 | 909835209 | 31-1287648 | 9-895580 |
| 911 | 829921 | 755826081 | 30-1827765 | 9-694069 | 970 | 940900 | 912637300 | 31-1448230 | 9-898983 |
| 912 | 831744 | 758392064 | 30-1993377 | 9-697615 | 971 | 942841 | 915448611 | 31-1608729 | 9-902383 |
| 913 | 833569 | 761018497 | 30-2158899 | 9-701158 | 972 | 944784 | 918300048 | 31-1769145 | 9-905782 |
| 914 | 835396 | 763655194 | 30-2324329 | 9-704499 | 973 | 946729 | 921167317 | 31-1929479 | 9-909178 |
| 915 | 837225 | 766303075 | 30-2489669 | 9-708227 | 974 | 948676 | 924010424 | 31-2089731 | 9-912571 |
| 916 | 839056 | 768957236 | 30-2654919 | 9-711772 | 975 | 950625 | 926859375 | 31-2249900 | 9-915962 |
| 917 | 840889 | 771092513 | 30-2820079 | 9-715305 | 976 | 952576 | 929714176 | 31-2409987 | 9-919351 |
| 918 | 842724 | 773626832 | 30-2985148 | 9-718835 | 977 | 954529 | 932574833 | 31-2569992 | 9-922738 |
| 919 | 844561 | 776161559 | 30-3150128 | 9-722363 | 978 | 956484 | 935441352 | 31-2729915 | 9-926122 |
| 920 | 846400 | 778696800 | 30-3315018 | 9-725888 | 979 | 958441 | 938313739 | 31-2889757 | 9-929504 |
| 921 | 848241 | 781232961 | 30-3479818 | 9-729411 | 980 | 960400 | 941192000 | 31-3049517 | 9-932884 |
| 922 | 850084 | 783774448 | 30-3644520 | 9-732931 | 981 | 962361 | 944076141 | 31-3209195 | 9-936261 |
| 923 | 851929 | 786320467 | 30-3809151 | 9-736448 | 982 | 964324 | 946966168 | 31-3368792 | 9-939636 |
| 924 | 853776 | 788870904 | 30-3973783 | 9-739963 | 983 | 966289 | 949862087 | 31-3528308 | 9-943009 |
| 925 | 855625 | 791423125 | 30-4138127 | 9-743476 | 984 | 968256 | 952763904 | 31-3687743 | 9-946380 |
| 926 | 857476 | 793976276 | 30-4302481 | 9-746986 | 985 | 970225 | 955671625 | 31-3847079 | 9-949748 |
| 927 | 859329 | 796530383 | 30-4466747 | 9-750493 | 986 | 972196 | 958585256 | 31-4006369 | 9-953114 |
| 928 | 861184 | 799085452 | 30-4630924 | 9-753998 | 987 | 974169 | 961504803 | 31-4165561 | 9-956477 |
| 929 | 863041 | 801641589 | 30-4795013 | 9-757500 | 988 | 976144 | 964430272 | 31-4324673 | 9-959839 |
| 930 | 864900 | 804207700 | 30-4959014 | 9-761000 | 989 | 978121 | 967361669 | 31-4483704 | 9-963198 |
| 931 | 866761 | 806774491 | 30-5122926 | 9-764497 | 990 | 980100 | 970299000 | 31-4642654 | 9-966555 |
| 932 | 868624 | 809351768 | 30-5286750 | 9-767992 | 991 | 982081 | 973242271 | 31-4801525 | 9-969909 |
| 933 | 870489 | 811929637 | 30-5450487 | 9-771484 | 992 | 984064 | 976191488 | 31-4960315 | 9-973262 |
| 934 | 872356 | 814508004 | 30-5614136 | 9-774974 | 993 | 986049 | 979146657 | 31-5119025 | 9-976612 |
| 935 | 874225 | 817086875 | 30-5777697 | 9-778462 | 994 | 988036 | 982107784 | 31-5277655 | 9-979960 |
| 936 | 876096 | 820025956 | 30-5941171 | 9-782946 | 995 | 990025 | 985074875 | 31-5436206 | 9-983306 |
| 937 | 877969 | 822526053 | 30-6104557 | 9-787429 | 996 | 992016 | 988047936 | 31-5594677 | 9-986649 |
| 938 | 879844 | 825026372 | 30-6267857 | 9-791909 | 997 | 994009 | 991026973 | 31-5753068 | 9-989990 |
| 939 | 881721 | 827526919 | 30-6431069 | 9-796386 | 998 | 996004 | 994011992 | 31-5911380 | 9-993329 |
| 940 | 883600 | 830027600 | 30-6594194 | 9-798861 | 999 | 998001 | 997002999 | 31-6069613 | 9-996666 |
| 941 | 885481 | 832527421 | 30-6757233 | 9-799334 | 1000 | 1000000 | 1000000000 | 31-6227766 | 10-000000 |

ANSWERS TO MISCELLANEOUS EXERCISES.

PAGES 58, 59.

2. Sixty-seven trillions eight hundred and forty-five billions three hundred and ninety-eight millions six hundred and seventy-eight thousand nine hundred and four.
Five quadrillions nine hundred trillions seven hundred and four billions sixty millions forty thousands, and sixty thousand six hundred and four *hundredths of millionths*.
3. $\overline{\text{M}}\overline{\text{V}}\overline{\text{DCC}}\overline{\text{LXIX}}$.
4. 429850000.
5. $\$67\cdot31\frac{1}{4}$.
6. 77991.
7. 605000070016·000000.
8. 46978900.
10. 69·800463.
11. ·8439.
12. 678900000.
13. 604329860000000.
14. 1000001000001001·000000000001.
15. ·0007609.
16. Ninety trillions eight hundred and seven billions sixty millions five hundred and four thousand and thirty.
Four quintillions four quadrillions forty trillions four hundred billions sixty thousand four hundred and thirty-two, and one trillion ten billion two hundred and three million forty thousand five hundred and six *hundredths of trillionths*.
18. 77 $\frac{1}{2}$ cords.
19. 717 cords 91 cubic feet.
20. $\overline{\text{DCCXVIII}}$, $\overline{\text{DCXIV}}$, $\overline{\text{CDXCIX}}$, $\overline{\text{CMXCIX}}$, $\overline{\text{VMMMDCXLIII}}$, $\overline{\text{XCVMCXLIX}}$, $\overline{\text{CLXMMMCMCLXXXVI}}$, $\overline{\text{CDXLMVCDXLIV}}$.
21. 333, 1989, and 1000001.
25. $\$3\cdot75\frac{1}{2}$, $\$24\cdot58\frac{1}{2}$, and $\$757\cdot47\frac{1}{2}$.

PAGES 100, 101.

- | | |
|---|---|
| <ol style="list-style-type: none"> 66. \$18029304. 67. \$13999999·73. 68. 36497318. 69. 35857536. 70. 27424500. 71. 271633. 72. 9504000. 73. 327040000. | <ol style="list-style-type: none"> 74. 92438 lbs. 8 oz. 2 dr. 1 scr. 13 grs. 75. 1698728602536. 76. 78990 bushels. 77. \$64·97. 78. 9032 yds. 3 qrs. 2 na. 79. 1037957691·5. 80. \$16444·9692. |
|---|---|

PAGES 116, 117.

- | | |
|---|---|
| 61. \$34736.8421. | 70. .578 oz. |
| 62. \$30634.9206. | 71. 50 $\frac{3}{4}$. |
| 63. 3308 dys. or 9 yrs. 20 $\frac{1}{2}$ dys. | 72. 250 lbs. |
| 64. \$32. | 73. 10.157. |
| 65. \$137. | 74. 2 bush. 1 pk. 1 gal. 2 qts. |
| 66. \$108. | 1 $\frac{1}{4}$ pts. |
| 67. \$9. | 75. 1898 $\frac{103}{100}$. |
| 68. \$29. | 76. 267 days 7 $\frac{1}{2}$ 55 $\frac{99}{100}$ hours. |
| 69. 429 $\frac{1}{3}$. | |

PAGES 118, 119, and 120.

- | | |
|---|--|
| 1. 789641420714. | 14. .0331632. |
| 2. Sixty-seven millions eight hundred and thirteen thousand four hundred and twenty, and twenty-one million thirty thousand and forty-six billionths. | 15. 475 $\frac{1}{4}$ hhd.. |
| Seventy-two millions, and seventy-two billionths. | 16. \$6750. |
| One billion one million and one hundred, and ten trillion ten million and one tenths of quadrillionths. | 17. 11 $\frac{1}{4}$. |
| 3. DCCIX, M $\overline{\text{VCCCLXXVI}}$, M $\overline{\text{XCMXCIX}}$, LXXXVMIV, MMCMXLVMMDXCVI. | 18. 58 acres. |
| 4. 53973 lbs. | 19. \$9.501. |
| 5. £3 18s. 11 $\frac{1}{2}$ d. | 20. \$37. |
| 6. 10837 yrs. 119 days 2 hours. | 21. 3 lbs. 0 oz. 14 dwt. 13 $\frac{1}{2}$ grs. |
| 7. \$2919.50 $\frac{1}{2}$. | 22. 29 acres 0 roods 21 per. |
| 8. \$123.77. | 23. 14 yds. |
| 9. 5200060020437000000005016. | 24. 15 lbs. 4 oz. 1 dwt. 14 grs. |
| 10. 1 acre 1 rood 3 per. 4 yds. 5 ft. 11 in. | 25. \$3890.38 $\frac{1}{2}$. |
| 11. \$12268.30. | 26. 1032694. |
| 12. 54 years 19 weeks, 3 days 16 hours 33 minutes. | 27. 16800. |
| 13. 741000000, '00741, 741000000, '000000741, '00000000741, '00741, and 741. | 28. \$360.15. |
| | 29. \$247.95. |
| | 30. \$132082. |
| | 31. 169.49 times. |
| | 32. \$79.99 $\frac{1}{2}$. |
| | 33. \$59.85. |
| | 34. \$532.12 $\frac{1}{2}$. |
| | 35. $\overline{\text{CCCCC}}\overline{\text{DCCIX}}$. |
| | 36. .56218+. |
| | 37. 1869696969.69. |
| | 38. \$1713.34. |
| | 39. \$21.1433. |
| | 40. 236 $\frac{1}{3}$. |

PAGES 149, 150.

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|--|---|
| 1. \$4688.16 $\frac{7}{8}$. | 4. 500313 <i>octenary</i> and 20222133 <i>quinary</i> . |
| 2. 27536 miles 1 fur. 21 per. 0 yds. 1 ft. 6 in. | 5. 1243994.98275. |
| 3. 96. | |

- | | |
|--|---|
| <p>6. $\overline{\text{LXXM}}\overline{\text{CDXXIII}}$ and $\overline{\text{CCXXXM}}\overline{\text{DLXVII}}$.</p> <p>7. 277200.</p> <p>8. See XLVIII Recapitulation Sec. I., page 57.</p> <p>9. 642762977065601.1.</p> <p>15. 742000000905000078014.0000087200011.</p> <p>16. Seventy-one trillions three hundred billions one hundred millions two hundred thousand four hundred and one, and seventy thousand four hundred and two trillionths.</p> <p>One hundred and thirty-four quadrillions nine hundred trillions one hundred and one billions one hundred thousand and one hundred, and two hundred million twenty thousand and two trillionths.</p> <p>Four quadrillions seven hundred trillion twenty thousand and seven, and two hundred and seventy-eight hundredths of trillionths.</p> <p>17. £2272 0s. 3½d.</p> | <p>10. —</p> <p>11. See Table, page 125.</p> <p>12. \$2689.51¾.</p> <p>13. 27.</p> <p>14. See Recapitulation XLVIII page 57.</p> <p>18. $2^5 \times 5^3 \times 3 \times 23$.</p> <p>19. 87 ft. 1' 1" 3''' 0'''' 10'''' 8'''' 10'''' 10''''''</p> <p>20. .011436.</p> <p>21. 16383.</p> <p>22. 4096.</p> <p>23. 11 acres 3 rds. 7 per. 19 yds. 0 ft. 130 in.</p> <p>24. 336960.</p> <p>25. Child's share, \$179.41¾; woman's, \$358.82⅙; man's \$179.412⅙.</p> <p>26. 1023 and 512.</p> <p>27. 99.47¾.</p> <p>28. 48359.8979694.</p> <p>29. 722487.0873859.</p> <p>30. 65 lbs. 7 oz. 0 drs. 1 scr.</p> <p>31. 1, 2, 4, 7, 8, 14, 19, 28, 38, 56, 76, 133, 152, 266, 532, 1064.</p> <p>32. 82½⅙ yards.</p> |
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PAGES 180, 181.

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|--|--|
| <p>1. $\frac{2}{3}$, $\frac{1}{100}$, $\frac{1}{20}$, $\frac{2}{25}$, and $\frac{7}{100}$.</p> <p>2. $\frac{2}{3}$.</p> <p>3. \$4.52½.</p> <p>4. $\frac{23}{36}$.</p> <p>5. Gave away $\frac{2}{10}$ and kept $\frac{10}{10}$.</p> <p>6. $1\frac{5}{8}$.</p> <p>7. \$212.99½.</p> <p>8. Longer part 72 feet and shorter part 64 feet.</p> <p>9. 1058½⅙ acres; \$13219.68½.</p> | <p>10. $14\frac{81}{10}$ and $\frac{27}{10}$.</p> <p>11. \$134.15½.</p> <p>12. \$28387.06½.</p> <p>13. $311\frac{8}{9}$ bushels.</p> <p>14. 1 and $1\frac{35}{75}$.</p> <p>15. $2\frac{1}{3}$ bushels.</p> <p>16. $\frac{1}{2}$.</p> <p>17. $4\frac{2}{3}$.</p> <p>18. $5\frac{2}{3}$ and $2\frac{2}{3}$.</p> <p>19. \$1333.33½ or $\frac{1}{10}$ of the whole</p> |
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PAGES 196, 197.

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|--|--|
| <p>84. .8.</p> <p>85. 1.44455667788.</p> <p>86. 4 days 17 hours 55 min. 30 sec.</p> <p>87. $1\frac{3}{4}\frac{2}{3}$.</p> | <p>88. 150.85931270094.</p> <p>89. .739157196 of a mile.</p> <p>90. 16 sq. ft. $104\frac{3}{4}$ inches.</p> <p>91. 1 acre 3 roods 13 per. 22 yds.</p> |
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408 ANSWERS TO MISCELLANEOUS EXERCISES.

- | | |
|--|---------------------------|
| 92. $11\frac{1}{6}$ and $1\frac{3}{6}$. | 97. $13\cdot5169533$. |
| 93. $26\cdot7837428571$. | 98. 3, 3, 1, 4, 1, and 9. |
| 94. $71\cdot86193$. | 99. $476\cdot65028119$. |
| 95. $11\cdot546$ oz. | 100. 9. |
| 96. $75\frac{1}{2}$ yards. | |

PAGES 198, 199, and 200.

- | | |
|---|--|
| 2. 702000007030017-0000000004000076. | 10. 20790. |
| 3. 1017116666 $\cdot\dot{6}$. | 11. $1375\dot{4}12$ and 2049151 |
| 4. $2\dot{3}$. | 12. 66. |
| 5. $10\cdot\frac{3837}{360}$. | 13. 1 day 23 hours 24 min. $34\frac{1}{2}$ |
| 6. 5044 bricks. | seconds. |
| 7. 111 sq. ft. 0' 9" 7''' 4'''' 5''''' | 14. 19860 lbs. 2 oz. $9\frac{1}{2}$ drs. |
| 8. $8\frac{1}{2}\frac{5}{16}$. | 15. \$158 \cdot 75. |
| 9. 12225 bush 2 pks 0 gal 2 qts. | 16. $\frac{8}{9}$, $\frac{7}{9}$, $\frac{24}{5}$, and $\frac{37}{10}$. |
| 17. 7040000, 0000704, 704000000000, 00000000704, 0000704, 704. | 17. 134501 $\frac{39}{9}$. |
| 18. $3\frac{20}{31}$. | 24. 13406 $\frac{1}{2}$ lbs. or 13406 $\frac{1}{2}$ gals. |
| 19. Man's share = £66 0s. $4\frac{1}{2}$ d., woman's = £33 0s. $2\frac{1}{2}$ d., child's = £11 0s. $0\frac{3}{4}$ d. | 25. \$295 \cdot 59 $\frac{7}{12}$. |
| 20. 100 $\frac{513}{30}$. | 26. 247 $\frac{1}{7}$. |
| 21. 1, 2, 3, 4, 5, 6, 9, 10, 12, 15, 18, 20, 25, 27, 30, 36, 45, 50, 54, 60, 75, 81, 90, 100, 108, 135, 150, 162, 180, 225, 270, 300, 324, 405, 450, 540, 675, 810, 900, 1350, 1620, 2025, 2700, 4050, 8100. | 27. 61 $\frac{2}{5}$. |
| 22. 117. | 28. — |
| 23. Lunar month = 29 days 12 hours 44 min. 3 seconds. Solar year = 365 days 5 hours 48 min. 48 seconds. | 29. — |
| | 30. $2^9 \times 3 \times 5$. |
| | 31. 55045884 lines. |
| | 32. \$45 \cdot 50. |
| | 33. \$90 \cdot 96 $\frac{3}{4}$. |
| | 34. $3\cdot185988$. |
| | 35. 2159 $\frac{1}{4}$. |
| | 36. \$21588 \cdot 90. |
| | 37. \$142 \cdot 8248. |
| | 38. 293. |
| | 39. $\frac{1410}{2310}$, $\frac{1848}{2310}$, $\frac{3316}{2310}$, $\frac{560}{2310}$, |
| | $\frac{1815}{2310}$, $\frac{417}{2310}$, $\frac{1155}{2310}$. |
| | 40. \$103 \cdot 35 $\frac{1}{2}$. |

PAGES 222, 223.

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|--|--------------------------------|
| 1. 2 : 3. | 4. Greatest 21:27; least 9:13. |
| 2. \$479 \cdot 30 $\frac{1}{2}$. | |
| 3. — | 5. 57-100555661872493. |
| 6. $53ce3\frac{7737}{4e7}$ duodenary, 12014313 $\frac{410042}{1033440}$ quinary, and 760f0 $\frac{9972}{12576}$ undenary. | |

ANSWERS TO MISCELLANEOUS EXERCISES. 409

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|---|--|
| 7. 5-57052 oz. | 14. $13\frac{2}{3}$. |
| 8. 3 yds. 3 qrs. 0 na. $0\frac{1}{4}$ in. | 15. £2 ls. $2\frac{1}{2}$ d. nearly. |
| 9. \$2962.70. | 16. $3\frac{1}{2}$ days. |
| 10. 1 bush. 2 pk. 0 gal. 1 qt. | 17. $\frac{1}{2}$. |
| 11. 17:8; 88:176; 17:8 and 23:11; 6:7 and 88:176; 1173:616. | 18. 52. |
| 12. 39 per cent. | 19. $50\frac{1}{2}$. |
| 13. $\frac{352}{2436}$. | 20. .026856599989+. |
| 23. 764876837 nonary; 10011110101000011001111010000 bi- nary; 11146453021 septenary. | 21. .0778. |
| 24. 188100. | 22. 432958 miles. |
| 25. $80\frac{1}{3}$. | 29. 1, 2, 3, 4, 5, 6, 7, 9, 10, 12, 14, 15, 18, 20, 21, 25, 28, 30, 35, 36, 42, 45, 50, 60, 63, 70, 75, 84, 90, 100, 105, 126, 140, 150, 175, 180, 210, 225, 252, 300, 315, 350, 420, 450, 525, 630, 700, 900, 1050, 1260, 1575, 2100, 3150, 6300. |
| 26. 48. | |
| 27. 415-471137804. | |
| 28. \$53-5966. | |
| 30. \$5.04. | |
| 31. Each man's share, \$325.99 $\frac{1}{3}$; each woman's, \$88.90 $\frac{1}{3}$; each child's, \$25.40 $\frac{1}{3}$. | |
| 32. $12\frac{1}{2}$, $5\frac{1}{2}$, $2\frac{1}{2}$. | 36. $\frac{2}{3}$. |
| 33. 3 yds. 2 ft. $8\frac{1}{2}$ in. | 37. $2\frac{1}{2}$. |
| 34. 104:5. | 38. 70 goats. |
| 35. 71 miles 5 fur. 34 per. 3 yards. | 39. 200. |

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- | | |
|---|---|
| 1. 7020400000, 7-0204, 70-204, 0000070204, 7020-4, and 00000070204. | 5. 5:7; 9:13; 54:221. |
| 2. 6704866-561. | 6. \$2070-3593. |
| 3. £399 19s. 5 $\frac{1}{2}$ d. | 7. They have none. |
| 4. 846-37295763. | 8. \$27431-314. |
| 12. 744916400000; 7-449164; 0007449164; 744916-4. | 9. 15, 55, 99, 147, and 195. |
| 13. — | 10. $2\frac{1}{2}$. |
| 14. Binary 63 and 32, Quater- nary 4095 and 1024, Sec- nary 46655 and 7776, Octe- nary 262143 and 32768, Duodenary 2985983 and 248832. | 11. 12 $\frac{1}{2}$ days. |
| | 15. 1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, 48, 54, 64, 72, 96, 108, 144, 192, 216, 288, 432, 576, 864, 1728. |
| | 16. 720720. |
| | 17. 70-789966677748855. |
| | 18. \$127-98. |
| | 19. 21-19117. |

ANSWERS TO EXAMINATION PROBLEMS.

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|---|--|
| 1. 700030000001900000001200006. | 20. 5456640. |
| 2. A, \$1639.32½; B, \$1528.21½; C, \$1437.31½; D, \$1534.95. | 21. They have none. |
| 3. 13½. | 22. A, \$3492.06; B, \$4761.91; C, \$6746.03. |
| 4. \$1497803819.4444. | 23. A, £167¼; B, £139¼; C, £93¼. |
| 5. 83160. | 24. 2½ hours. |
| 6. 361 y'rs. 10 m'ths. 25 days. | 25. LXXXVCMXXXVIII and XVMMCDXCVMMDCLXXIX. |
| 7. 40.38. | 26. 1st gets 792 loaves; 2nd, 594; 3rd, 924. |
| 8. 33943 lbs. 4 oz. 8 dwt. 14½ grs. | 27. 72, 18 and 54 lbs., or 24, 96, and 96 lbs. respectively. |
| 9. 2. | 28. \$3725.764. |
| 10. 120½. | 29. 24010.23. |
| 11. 3. | 30. \$4803.5064. |
| 12. 24. | 31. 5739.29 yds. Gain 25½ per cent. |
| 13. A, \$384.47; B, \$291.07; C, \$221.89. | 32. — |
| 14. 135½ lbs. | 33. \$126.12. |
| 15. 165229. | |
| 16. 530.00121864500. | 40. \$460.0034. |
| 17. \$7854.29. | 41. 5 yrs. 8 mos. 5 days. |
| 18. 26½. | 42. Amount \$1409.07. Com- pound Int. \$595.36. |
| 19. 81000. | 43. 10 months 18 days. |
| 34. 2.886057; 1.290035; 3.051153; 1.449735; 4.812913; 4.698970; 2.182129; 0.909128. | |
| 35. 18.12. | |
| 36. 84 years. | |
| 37. 66.80578 times. | |
| 38. 22992700.72992700. | |
| 39. \$5.482. | |
| 44. A, \$571.9675; B, \$554.8675; C, \$535.6375; D, \$493.5275; and E, \$1078. | |
| 45. \$1372.02898. | Seven trillions six hundred billions two hundred and ninety millions thirty- four thousand and seven, and sixty-seven millions four hundred thousand two hundred and nine quadrillionths. |
| 46. 1. | 50. 1296. |
| 47. 117042723744337 octenary. | 51. 33.395 years. |
| 48. .01 and .012345679. | |
| 49. One quadrillion three hundred billions fifty mil- lion and six thousand, and seven hundred mil- lion eighty thousand and nine trillionths. | |

52. $7119\frac{2}{9}$.
 53. 144.
 54. $35\frac{2}{3}$.
 55. $8\frac{1}{2}$ days.
 56. \$2469.71.
 57. $4\frac{2}{13}$, $3\frac{1}{13}$, and $2\frac{1}{13}$.
 58. Each man had 60; A caught 50, B 60, C 70.
 59. 191 and 17763.
 60. 44.997 years.
 61. A, \$1556.95 $\frac{1}{2}$; B, \$1169.05 $\frac{2}{3}$; C, \$973.08 $\frac{1}{3}$.
 62. 1, 2, 4, 1429, 2858, 5716.
 63. $2\frac{2}{5}$.
 64. Man's share = \$919.14 $\frac{1}{2}$;
 Woman's = \$459.57 $\frac{1}{2}$;
 and child's = \$153.19 $\frac{1}{2}$.
 65. 24.
 66. \$21.03.
 67. Greatest 9:16; least 10:19;
 comp. ratio 21 : 247.
 68. 6.585461; 3.502675; 5.187521; 2.113509; 0.196295;
 1.969276.
 69. \$4.314.
 70. X \$672 and Y \$1120.
 71. $\frac{2}{3}$.
 72. 4321.
 73. $18\frac{2}{3}$ lbs. at 4d.; $18\frac{2}{3}$ lbs. at 6d.; and $74\frac{2}{3}$ lbs. at 8d.
 74. 10, 22, 26.
 75. 11000000000011.0000000011.
 76. \$3649.3932.
 77. $2^8 \times 3^2 \times 7 \times 11$.
 78. $28\frac{1}{2}$.
 79. A, £194 16s. $1\frac{1}{2}$ d.; B, £129 17s. $4\frac{1}{2}$ d.; C, £97 8s. $0\frac{1}{2}$ d.; D, £77 18s. $5\frac{1}{2}$ d.
 80. \$1230.338.
 81. 10 hours.
 82. 41 years.
 83. 4.629 days.
 84. £4 16s.
 85. $44\frac{1}{2}$.
 86. 1422.2 lbs.
 87. 1st, \$920.20; 2nd, \$2760.60; 3rd, \$5521.20.
 88. Paid each workman \$22.66 $\frac{1}{2}$; 1st company cleared $87\frac{1}{2}$ acres; 2d company, $77\frac{1}{2}$ acres; cost of clearing, \$8.45 $\frac{1}{2}$ per acre.
 89. 8.5318452.
 90. 0.019156118.
 91. 2781.850714957599739858.
 92. 157.036 feet.
 93. 85 spirits, 35 water.
 94. 422.32.
 95. 70 and 14.
 96. 223.82469585.
 97. 5.22341.
 98. 58 and 28.
 99. 156240.
 100. 30401.
 101. $228\frac{1}{2}$; 1617.
 102. 3 and $1\frac{1}{2}$, or 4 and $1\frac{1}{2}$, or 5 and $1\frac{1}{2}$, &c.
 103. $1\frac{18}{17}$.
 104. $5\frac{1}{17}$ minutes past 1 o'clock.
 105. 1, $8\frac{1}{2}$, $16\frac{1}{2}$, $24\frac{1}{2}$, $32\frac{1}{2}$, 40.
 106. 7.
 107. Apple 2d., pear 3d.
 108. $\frac{1}{2}$.
 109. \$275.
 110. \$124 and \$1564.
 111. 117.
 112. $62\frac{1}{2}$ gal., $83\frac{1}{2}$ gal., and 146 gal.
 113. 1st, .47 inches; 2nd, .57 in.; 3rd, .82 in.; 4th, 3.149 in.
 114. 71-117.
 115. \$2019.651; \$4871.803;
 \$4815.805; \$6467.739;
 \$1825.
 116. 1st 300 yrs; 2nd 56.827 yrs.

412 ANSWERS TO EXAMINATION PROBLEMS.

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|--|--|
| 117. 15 and 11. | 131. 8·04 in. 9·534 in. 12·426 in. and 30 inches. |
| 118. \$2340·00. | 132. 51 of each, rem. $\text{£}1\frac{1}{10}$. |
| 119. 132 days. | 133. \$200. |
| 120. A, \$2180 ; B, \$1635 ; C, \$1308 ; D, \$1090. | 134. 11 per cent. nearly. |
| 121. $7, 43, 237, 1166\frac{2}{3}, 8166\frac{2}{3}$. | 135. \$1388·888. |
| 122. $801\frac{2}{3}$ and $411, 302\frac{2}{3}$. | 136. 1s. 9d., 1s. 2d., and 7d. |
| 123. Sum $\text{£}58$ os. 8 $\frac{1}{2}$ d.; quotient 32414·56. | 137. A, \$25 ; B, \$25 ; C, \$50 ; D, \$100. |
| 124. $491\frac{1}{100}$ yds. | 138. ·057. |
| 125. \$214. | 139. $\frac{32}{787}$; $162\frac{22}{110}$; $1\frac{21}{13}$; $\frac{54}{233}$; 2308. |
| 126. 1 st 175 yrs.; 2 nd 41·914 yrs. | 140. 96 ; 17 $\frac{1}{2}$. |
| 127. $10\frac{1}{37}$ perches. | 141. \$89 $\frac{1}{3}$; \$107 $\frac{1}{2}$; \$143 $\frac{2}{3}$; and \$179 $\frac{2}{5}$. |
| 128. 111104. | 142. \$15009·84. |
| 129. 9, 27, 81, 243, 729, 2187, 6561. | 143. 17 $\frac{1}{2}$, 32 $\frac{1}{2}$, 48 $\frac{1}{2}$ and 63 $\frac{1}{2}$; 35 and 85905. |
| 130. 9 $\frac{1}{2}$. | 144. 36 $\frac{1}{2}$ days. |

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